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TYPIS ET IN AEDIBUS B. G. TEUBNERI
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LEONHARDI EULERI
COMMENTATIONES ANALYTICAE
AD THEORIAM INTEGRALIUM ELLIPTICORUM PERTINENTES

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ADOLF KRAZER

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506, 581, 582, 590, 605, 624, 633, 638, 639, 645, 676, 714, 780, 781, 782, 783, 817, 818, 819

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QUAE TOMIS 20 ET 21 CONTINENTUR

DILUCIDATIONES SUPER METHODO ELEGANTISSIMA QUA ILLUSTRIS DE LA GRANGE USUS EST IN INTEGRANDA AEQUATIONE DIFFERENTIALI $\frac{dx}{\sqrt{X}} = \frac{dy}{\sqrt{Y}}$

Commentatio 506 indicis ENESTROEMIANI
Acta academiae scientiarum Petropolitanae 1778: I (1780), p. 20—57

1. Postquam diu et multum in perscrutanda aequatione differentiali

$$\frac{dx}{\sqrt{X}} = \frac{dy}{\sqrt{Y}}$$

desudassem atque imprimis in methodum *directam*, quae via facili ac plana ad eius integrale perduceret, nequicquam inquisivissem, penitus obstupui, cum mihi nunciaretur in volumine quarto Miscellaneorum Taurinensium ab Illustri DE LA GRANGE[1]) talem methodum esse expositam, cuius ope pro casu, quo

$$X = A + Bx + Cxx + Dx^3 + Ex^4$$

et

$$Y = A + By + Cyy + Dy^3 + Ey^4,$$

propositae aequationis differentialis hoc integrale algebraicum atque adeo

1) I. L. LAGRANGE, *Sur l'intégration de quelques équations différentielles dont les indéterminées sont séparées, mais dont chaque membre en particulier n'est point intégrable*, Misc. Taur. 4 (1766/9): II, p. 98; *Oeuvres de LAGRANGE*, publiées par les soins de M. I.-A. SERRET, t. II, p. 5. A. K.

completum felicissimo successu elicuit

$$\frac{\sqrt{X}+\sqrt{Y}}{x-y} = \sqrt{(A + D(x+y) + E(x+y)^2)},$$

ubi A denotat quantitatem constantem arbitrariam per integrationem ingressam.

2. Istud autem egregium inventum eo magis sum admiratus, quod equidem semper putaveram talem methodum in investigando idoneo factore, quo aequatio proposita integrabilis redderetur, quaeri oportere, cum vulgo omnis methodus integrandi vel in separatione variabilium vel in idoneo multiplicatore contineri videatur, etiamsi certis casibus quoque ipsa differentiatio ad integrale perducere queat, quemadmodum tam a me quam aliis plurima exempla est ostensum. Ad hanc autem tertiam viam illa ipsa methodus GRANGIANA rite referri posse videtur.

3. Quanquam autem facile est inventis aliquid addere, tamen in re tam ardua plurimum intererit hanc methodum ab Illustri LA GRANGE adhibitam accuratius perpendisse atque ad usum analyticum magis accommodasse, ut quidem totum negotium multo facilius ac simplicius expediri posse videatur; quamobrem, quae de hoc argumento, quod merito maximi momenti est censendum, sum meditatus, hic data opera fusius sum expositurus.

4. Quoniam autem hoc integrale ab Illustri LA GRANGE inventum ab iis formulis, quas ipse olim dederam, plurimum discrepat ac simplicitate non mediocriter antecellit, ante omnia visum est scrutari, quomodo aequationi differentiali satisfaciat. Hunc in finem pono brevitatis gratia $\sqrt{X} + \sqrt{Y} = V$, ut habeam

$$\frac{V}{x-y} = \sqrt{(A + D(x+y) + E(x+y)^2)},$$

quam aequationem ita differentiare oporteat, ut constans arbitraria A ex differentiali excedat. Sumtis igitur quadratis erit

$$\frac{V^2}{(x-y)^2} = A + D(x+y) + E(x+y)^2.$$

quae differentiata dat

$$\frac{2VdV}{(x-y)^2} - \frac{2VV(dx-dy)}{(x-y)^3} - D(dx+dy) - 2E(x+y)(dx+dy) = 0.$$

5. Quo nunc calculus planior reddatur, seorsim partes vel per dx vel per dy affectas investigemus. Pro elemento igitur dx, si y ut constans spectetur, erit

$$dV = \frac{X'dx}{2\sqrt{X}},$$

unde singulae partes ita se habebunt

$$dx\left(\frac{VX'}{(x-y)^2\sqrt{X}} - \frac{2VV}{(x-y)^3} - D - 2E(x+y)\right),$$

ubi notetur esse $V = \sqrt{X} + \sqrt{Y}$ hincque

$$VV\sqrt{X} = (X+Y)\sqrt{X} + 2X\sqrt{Y},$$

unde hic duplicis generis termini occurrunt, dum vel per $\sqrt{X}$ vel per $\sqrt{Y}$ sunt affecti. Duo autem termini adsunt $\sqrt{Y}$ affecti, qui sunt

$$-\frac{4X\sqrt{Y}}{(x-y)^3} + \frac{X'\sqrt{Y}}{(x-y)^2},$$

qui ergo iunctim sumti dabunt

$$\frac{\sqrt{Y}}{(x-y)^3}(X'(x-y) - 4X),$$

quae forma ob

$$X = A + Bx + Cxx + Dx^3 + Ex^4$$

hincque

$$X' = B + 2Cx + 3Dxx + 4Ex^3$$

dabit

$$X'(x-y) - 4X = -4A - B(3x+y) - 2C(xx+xy) - D(x^3+3xxy) - 4Ex^3y.$$

Termini autem per $\sqrt{X}$ affecti sunt

$$\frac{\sqrt{X}}{(x-y)^3}\left(X'(x-y) - 2(X+Y) - D(x-y)^3 - 2E(x+y)(x-y)^3\right).$$

completum felicissimo successu elicuit

$$\frac{\sqrt{X}+\sqrt{Y}}{x-y}=\sqrt{(\varDelta+D(x+y)+E(x+y)^2)},$$

ubi $\varDelta$ denotat quantitatem constantem arbitrariam per integrationem ingressam.

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$$\frac{V}{x-y}=\sqrt{(\varDelta+D(x+y)+E(x+y)^2)},$$

quam aequationem ita differentiare oportet, ut constans arbitraria $\varDelta$ ex differentiali excedat. Sumtis igitur quadratis erit

$$\frac{V^2}{(x-y)^2}=\varDelta+D(x+y)+E(x+y)^2,$$

quae differentiata dat

$$\frac{2VdV}{(x-y)^2} - \frac{2VV(dx-dy)}{(x-y)^3} - D(dx+dy) - 2E(x+y)(dx+dy) = 0.$$

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$$dV = \frac{X'dx}{2\sqrt{X}},$$

unde singulae partes ita se habebunt

$$dx\left(\frac{VX'}{(x-y)^2\sqrt{X}} - \frac{2VV}{(x-y)^3} - D - 2E(x+y)\right),$$

ubi notetur esse $V = \sqrt{X} + \sqrt{Y}$ hincque

$$VV\sqrt{X} = (X+Y)\sqrt{X} + 2X\sqrt{Y},$$

unde hic duplicis generis termini occurrunt, dum vel per $\sqrt{X}$ vel per $\sqrt{Y}$ sunt affecti. Duo autem termini adsunt $\sqrt{Y}$ affecti, qui sunt

$$-\frac{4X\sqrt{Y}}{(x-y)^3} + \frac{X'\sqrt{Y}}{(x-y)^2},$$

qui ergo iunctim sumti dabunt

$$\frac{\sqrt{Y}}{(x-y)^3}(X'(x-y) - 4X),$$

quae forma ob

$$X = A + Bx + Cxx + Dx^3 + Ex^4$$

hincque

$$X' = B + 2Cx + 3Dxx + 4Ex^3$$

dabit

$$X'(x-y) - 4X = -4A - B(3x+y) - 2C(xx+xy) - D(x^3+3xxy) - 4Ex^3y.$$

Termini autem per $\sqrt{X}$ affecti sunt

$$\frac{\sqrt{X}}{(x-y)^3}\left(X'(x-y) - 2(X+Y) - D(x-y)^3 - 2E(x+y)(x-y)^3\right).$$

Cum igitur sit

$$X+Y=2A+B(x+y)+C(x^2+y^2)+D(x^3+y^3)+E(x^4+y^4),$$

facta substitutione iste postremus factor erit

$$-4A-B(x+3y)-2C(xy+yy)-D(3xyy+y^3)-4Exy^3,$$

quae forma a praecedente hoc tantum discrepat, quod litterae x et y sunt permutatae.

6. Quod si ergo brevitatis gratia ponamus

$$M=4A+B(3x+y)+2C(xx+xy)+D(x^3+3xxy)+4Ex^3y,$$
$$N=4A+B(x+3y)+2C(yy+xy)+D(y^3+3xyy)+4Exy^3,$$

hinc pars elemento dx affecta ita erit expressa

$$-\frac{dx}{(x-y)^3\sqrt{X}}(M\sqrt{Y}+N\sqrt{X}).$$

7. Simili modo ob

$$dV=\frac{Y'dy}{2\sqrt{Y}}$$

partes elemento dy affectae erunt

$$\frac{dy}{\sqrt{Y}}\left(\frac{VY'}{(x-y)^2}+\frac{2VV\sqrt{Y}}{(x-y)^3}-D\sqrt{Y}-2E(x+y)\sqrt{Y}\right).$$

Haec iam forma ob

$$V=\sqrt{X}+\sqrt{Y} \quad \text{et} \quad VV\sqrt{Y}=(X+Y)\sqrt{Y}+2Y\sqrt{X}$$

sequentes terminos per $\sqrt{X}$ affectos

$$\frac{\sqrt{X}}{(x-y)^3}(Y'(x-y)+4Y),$$

quae forma ex priore praecedentis calculi oritur, si litterae x et y permutentur simulque signa; unde patet hanc expressionem praebere valorem $+N$.

Reliqui autem termini per $\sqrt{Y}$ effecti erunt

$$\frac{\sqrt{Y}}{(x-y)^3}\left(Y'(x-y)+2(X+Y)-D(x-y)^3-2E(x+y)(x-y)^3\right).$$

Haec forma iterum ex permutatione litterarum et signorum ex forma praecedentis calculi oritur; quae ergo cum esset $-N$, haec erit $+M$. Hoc igitur modo partes elementum dy continentes erunt

$$+\frac{dy}{(x-y)^3\sqrt{Y}}(N\sqrt{X}+M\sqrt{Y}).$$

8. Coniungendis igitur his membris aequatio differentialis ex forma GRANGIANA orta erit

$$\left(\frac{dy}{\sqrt{Y}}-\frac{dx}{\sqrt{X}}\right)\frac{N\sqrt{X}+M\sqrt{Y}}{(x-y)^3}=0,$$

quae per factorem communem divisa praebet ipsam aequationem differentialem propositam $\frac{dx}{\sqrt{X}}=\frac{dy}{\sqrt{Y}}$; unde simul patet aequationem integralem exhibitam recte se habere atque adeo valorem litterae $\varDelta$ arbitrio nostro penitus relinqui.

9. Antequam autem methodum GRANGIANUM ad ipsam aequationem differentialem $\frac{dx}{\sqrt{X}}=\frac{dy}{\sqrt{Y}}$ in omni extensione acceptam applicemus, a casu simpliciore inchoemus, quo aequatio adeo rationalis proponitur haec

$$\frac{dx}{a+2bx+cxx}=\frac{dy}{a+2by+cyy}.$$

ANALYSIS
PRO INTEGRATIONE AEQUATIONIS DIFFERENTIALIS

$$\frac{dx}{a+2bx+cxx}=\frac{dy}{a+2by+cyy}$$

10. Ponamus brevitatis gratia

$$a+2bx+cxx=X$$

et

$$a+2by+cyy=Y,$$

ut fieri debeat

$$\frac{dx}{X} = \frac{dy}{Y};$$

quae formulae cum inter se debeant esse aequales, utraque per idem elementum dt designetur, ita ut nanciscamur has duas formulas

$$\frac{dx}{dt} = X \quad \text{et} \quad \frac{dy}{dt} = Y.$$

Quodsi ergo iam statuamus $x - y = q$, erit

$$\frac{dq}{dt} = X - Y = 2bq + cq(x + y),$$

unde per q dividendo erit

$$\frac{dq}{qdt} = 2b + c(x + y).$$

11. Nunc primas formulas differentiemus sumto elemento dt constante et facto

$$dX = X'dx \quad \text{et} \quad dY = Y'dy$$

orientur hae duae aequationes

$$\frac{ddx}{dxdt} = X' \quad \text{et} \quad \frac{ddy}{dydt} = Y',$$

quae invicem additae praebent

$$\frac{ddx}{dxdt} + \frac{ddy}{dydt} = X' + Y'.$$

Quare, cum sit

$$X' = 2b + 2cx \quad \text{et} \quad Y' = 2b + 2cy,$$

erit

$$\frac{1}{dt}\left(\frac{ddx}{dx} + \frac{ddy}{dy}\right) = 4b + 2c(x + y).$$

oniam igitur hic postremus valor duplo maior est praecedente $\frac{dq}{qdt}$, ıducti sumus ad hanc aequationem

$$\frac{ddx}{dx} + \frac{ddy}{dy} = \frac{2dq}{q},$$

quae integrata dat $ldx + ldy = 2lq +$ const., hincque in numeris erit

$$dxdy = Cqqdt^2,$$

ita ut sit

$$C = \frac{dxdy}{qqdt^2}.$$

Quare, cum sit

$$\frac{dx}{dt} = X \quad \text{et} \quad \frac{dy}{dt} = Y,$$

aequatio integralis erit

$$\frac{XY}{(x-y)^2} = C,$$

quae ergo non solum est algebraica, sed etiam completa.

13. Si igitur proposita fuerit haec aequatio differentialis

$$\frac{dx}{a + 2bx + cxx} = \frac{dy}{a + 2by + cyy},$$

eius integrale completum ita erit expressum

$$\frac{(a + 2bx + cxx)(a + 2by + cyy)}{(x-y)^2} = C,$$

quae utrinque addendo $bb - ac$ induet hanc formam

$$\frac{aa + 2ab(x+y) + 2acxy + bb(x+y)^2 + 2bcxy(x+y) + ccxxyy}{(x-y)^2} = \varDelta\varDelta,$$

sicque extracta radice integrale hanc formam habebit

$$\frac{a + b(x+y) + cxy}{x-y} = \varDelta,$$

quae sine dubio est simplicissima, quandoquidem tam y per x quam x per y facillime exprimi potest, cum sit

$$y = \frac{(\varDelta - b)x - a}{\varDelta + b + cx} \quad \text{et} \quad x = \frac{a + (\varDelta + b)y}{\varDelta - b - cy}.$$

14. Calculum, quo hic usi sumus, perpendenti facile patebit in his formis X et Y non ultra quadrata progredi licere. Si enim ipsi X insuper tri-

buamus terminum dx^3 et ipsi Y terminum dy^3, pro priore forma prodit

$$\frac{X-Y}{x-y} = 2b + c(x+y) + d(xx+xy+yy) = \frac{dq}{qdt},$$

pro altera autem forma est

$$X' + Y' = 4b + 2c(x+y) + 3d(xx+yy) = \frac{ddx}{dxdt} + \frac{ddy}{dydt}.$$

Quare si hinc duplum praecedentis auferamus, colligitur

$$\frac{ddx}{dxdt} + \frac{ddy}{dydt} - \frac{2dq}{qdt} = d(x-y)^2,$$

quam aequationem non amplius integrare licet.

15. Facile autem ostendi potest talem aequationem differentialem, in qua ultra quadratum proceditur, nullo amplius modo algebraico integrari posse. Si enim tantum hic casus proponeretur $\frac{dx}{1+x^3} = \frac{dy}{1+y^3}$, notum est utrinque integrale partim logarithmos partim arcus circulares involvere ideoque quantitates transcendentes diversos, quae nullo modo inter se comparari possunt. Huiusmodi scilicet comparationes iis tantum casibus locum habere possunt, quando utrinque unius generis tantum quantitates transcendentes occurrunt.

ANALYSIS
PRO INTEGRATIONE AEQUATIONIS

$$\frac{dx}{a+2bx+cxx} + \frac{dy}{a+2by+cyy} = 0$$

16. Quodsi hic ut ante ponamus

$$\frac{dx}{a+2bx+cxx} = dt,$$

statui debebit

$$\frac{dy}{a+2by+cyy} = -dt;$$

at vero si calculum simili modo quo ante instituere velimus, nihil plane proficimus. Postquam autem omnes difficultates probe perpendissem, tandem in artificium incidi, quo hunc casum expedire licuit, ita ut hinc non contemnendum incrementum methodo GRANGIANAE attulisse mihi videar.

17. Quoniam igitur has duas habeo aequationes

$$\frac{dx}{dt} = X \quad \text{et} \quad \frac{dy}{dt} = -Y,$$

hinc formo istam novam aequationem

$$\frac{y\,dx + x\,dy}{dt} = yX - xY.$$

Iam facio $xy = u$, ut habeam

$$\frac{du}{dt} = a(y - x) + cxy(x - y),$$

unde posito $x - y = q$ erit $\frac{du}{dt} = q(cu - a)$, quae aequatio per $cu - a$ divisa ductaque in c praebet

$$\frac{c\,du}{dt(cu - a)} = cq,$$

hocque modo nacti sumus differentiale logarithmicum.

18. Dein vero aequationes principales ut ante differentiemus et obtinebimus

$$\frac{ddx}{dt\,dx} = X' \quad \text{et} \quad \frac{ddy}{dt\,dy} = -Y',$$

quae invicem additae dant

$$\frac{1}{dt}\left(\frac{ddx}{dx} + \frac{ddy}{dy}\right) = X' - Y' = 2cq;$$

quare si hinc duplum praecedentis aequationis subtrahamus, remanebit

$$\frac{1}{dt}\left(\frac{ddx}{dx} + \frac{ddy}{dy} - \frac{2c\,du}{cu - a}\right) = 0,$$

unde per dt multiplicando et integrando nanciscimur $l\,dx + l\,dy - 2l(cu - a) = lC$ ideoque $\frac{dx\,dy}{(cu - a)^2} = C\,dt^2$. Cum igitur sit $dx = X\,dt$ et $dy = -Y\,dt$, aequatio integralis nostra erit $-\frac{XY}{(cu - a)^2} = C$.

19. Per hanc ergo analysin deducti sumus ad hanc aequationem integralem aequationis propositae

$$\frac{(a+2bx+cxx)(a+2by+cyy)}{(a-cxy)^2}=C,$$

quae aequatio, si utrinque unitas subtrahatur, reducitur ad hanc formam

$$\frac{2ab(x+y)+ac(x+y)^2+4bbxy+2bcxy(x+y)}{(a-cxy)^2}=C.$$

20. Illustremus hanc integrationem exemplo, ponendo $a=1$, $b=0$ et $c=1$, ita ut proposita sit haec aequatio differentialis

$$\frac{dx}{1+xx}+\frac{dy}{1+yy}=0,$$

cuius integrale novimus esse $\text{A tang. } x+\text{A tang. } y=\text{A tang. } \frac{x+y}{1-xy}=C$, sicque novimus esse $\frac{x+y}{1-xy}=C$. At vero nostra postrema formula dat pro hoc casu

$$\frac{(x+y)^2}{(1-xy)^2}=C \quad \text{ideoque} \quad \frac{x+y}{1-xy}=C,$$

quod egregie convenit.

21. Consideremus etiam casum, quo $a=1$, $b=\frac{1}{2}$ et $c=1$, ita ut proponatur haec aequatio

$$\frac{dx}{1+x+xx}+\frac{dy}{1+y+yy}=0,$$

cuius integrale est

$$\frac{2}{\sqrt{3}}\text{ A tang. }\frac{x\sqrt{3}}{2+x}+\frac{2}{\sqrt{3}}\text{ A tang. }\frac{y\sqrt{3}}{2+y}=C,$$

unde sequitur fore

$$\text{A tang. }\frac{2(x+y+xy)\sqrt{3}}{4+2(x+y)-2xy}=C$$

ideoque etiam $\frac{x+y+xy}{2+x+y-xy}=C$. At vero forma integralis inventa pro hoc casu dabit

$$\frac{x+y+(x+y)^2+xy+xy(x+y)}{(1-xy)^2}=C,$$

quae in factores resoluta dat

$$\frac{(1+x+y)(x+y+xy)}{(1-xy)^2} = C.$$

Prior vero aequatio $\frac{x+y+xy}{2+x+y-xy} = C$ inversa praebet $\frac{2+x+y-xy}{x+y+xy} = C$ et unitate subtracta $\frac{1-xy}{x+y+xy} = C$ atque haec in praecedentem ducta dat $\frac{1+x+y}{1-xy} = C$.

22. Videamus igitur, utrum hae posteriores aequationes inter se conveniant, et quia constantes utrinque inter se discrepare possunt, ambas aequationes ita referamus

$$\frac{1-xy}{x+y+xy} = \alpha \quad \text{et} \quad \frac{1+x+y}{1-xy} = \beta;$$

unde cum sit $\frac{1}{\alpha} = \frac{x+y+xy}{1-xy}$, evidens est fore $\beta - \frac{1}{\alpha} = 1$, ex quo pulcherrimus consensus inter ambas formulas elucet.

Ex his exemplis intelligitur aequationem generalem supra inventam hoc modo per factores repraesentari posse

$$\frac{(2b+c(x+y))(a(x+y)+2bxy)}{(a-cxy)^2}.$$

Ceterum consideratio harum formularum haud iniucundas speculationes suppeditare poterit.

23. Sequenti autem modo forma illa integralis inventa

$$\frac{(2b+c(x+y))(a(x+y)+2bxy)}{(a-cxy)^2} = C$$

statim ad formam simplicissimam reduci potest; si enim eius factores statuamus

$$\frac{2b+c(x+y)}{a-cxy} = P \quad \text{et} \quad \frac{a(x+y)+2bxy}{a-cxy} = Q,$$

ut esse debeat $PQ = C$, erit $aP - cQ = \frac{2ab-2bcxy}{a-cxy} = 2b$, unde fit

$$Q = \frac{aP-2b}{c},$$

sicque quantitati constanti aequari debet haec forma $\frac{aPP-2bP}{c}$; ex quo patet, etiam ipsam quantitatem P constanti aequari debere, ita ut iam aequatio nostra integralis sit

$$\frac{2b+c(x+y)}{a-cxy}=C \quad \text{vel etiam} \quad \frac{a(x+y)+2bxy}{a-cxy}=C.$$

ALIA SOLUTIO FACILLIMA EIUSDEM AEQUATIONIS

$$\frac{dx}{a+2bx+cxx}+\frac{dy}{a+2by+cyy}=0$$

24. Postrema reductione probe perpensa comperui statim ab initio ad formam integralis simplicissimam perveniri posse atque adeo non necesse esse ad differentialia secunda ascendere. Si enim ut ante ponamus $x+y=p$, $x-y=q$ et $xy=u$, ex formulis

$$\frac{dx}{dt}=X \quad \text{et} \quad \frac{dy}{dt}=-Y$$

statim deducimus $\frac{dp}{dt}=X-Y=2bq+cpq$, unde fit

$$\frac{dp}{2b+cp}=qdt.$$

25. Porro vero erit

$$\frac{ydx+xdy}{dt}=\frac{du}{dt}=yX-xY=-aq+cqu,$$

unde fit $\frac{du}{cu-a}=qdt$, quam ob rem hinc statim colligimus hanc aequationem $\frac{dp}{2b+cp}=\frac{du}{cu-a}$, cuius integratio praebet $l(2b+cp)=l(cu-a)+lC$; unde deducitur haec aequatio algebraica $\frac{2b+cp}{cu-a}=C$, quae restitutis litteris x et y dat $\frac{2b+c(x+y)}{cxy-a}=C$, quae est forma simplicissima aequationis integralis desi-

Hic imprimis notatu dignum occurrit, quod casum primum hac ...solvere non licet.

inventa facile aliae derivantur; veluti si $\frac{\ldots y)+2bxy}{\ldots xy-a}=C$, quae per praecedentem di- ...at, scilicet $\frac{2b+c(x+y)}{a(x+y)+2bxy}=C$; quae formae

quomodo satisfaciant, operae pretium erit ostendisse. Et quidem postrema forma, differentiata, erit

$$\frac{-2ab(dx+dy)-4bb(ydx+xdy)-2bc(yydx+xxdy)}{(a(x+y)+2bxy)^2},$$

quae in ordinem redacta praebet

$$dx(2ab+4bby+2bcyy)+dy(2ab+4bbx+2bcxx)=0.$$

Haec per $2b$ divisa et separata dat

$$\frac{dx}{a+2bx+cxx}+\frac{dy}{a+2by+cyy}=0,$$

quae est ipsa proposita.

ANALYSIS
PRO INTEGRATIONE AEQUATIONIS
$$\frac{dx}{\sqrt{(A+Bx+Cxx)}}=\frac{dy}{\sqrt{(A+By+Cyy)}}$$

27. Introducto novo elemento dt, deinceps pro constanti habendo, oriuntur hae duae aequationes

$$\frac{dx}{dt}=\sqrt{X} \quad \text{et} \quad \frac{dy}{dt}=\sqrt{Y},$$

ubi litteris X et Y valores initio assignatos tribuamus. Videbimus autem pro methodo, qua hic utemur, terminos litteris D et E affectos omitti debere. Sumtis ergo quadratis erit

$$\frac{dx^2}{dt^2}=X \quad \text{et} \quad \frac{dy^2}{dt^2}=Y.$$

28. Nunc istas formulas differentiemus positoque, ut fieri solet, $dX=X'dx$ et $dY=Y'dy$ nanciscemur has aequationes

$$\frac{2ddx}{dt^2}=X' \quad \text{et} \quad \frac{2ddy}{dt^2}=Y'$$

ac posito $x+y=p$ fiet $\frac{2ddp}{dt^2}=X'+Y'$. Cum iam sit

$$X'=B+2Cx+3Dxx+4Ex^3 \quad \text{et} \quad Y'=B+2Cy+3Dyy+4Ey^3,$$

erit

$$X'+Y'=2B+2Cp+3D(xx+yy)+4E(x^3+y^3)=\frac{2ddp}{dt^2},$$

quae aequatio manifesto integrationem admittet, si fuerit et $D=0$ et $E=0$, quemadmodum assumsimus. Multiplicando igitur per dp et integrando nanciscimur

$$\frac{dp^2}{dt^2}=\varDelta+2Bp+Cpp$$

et radicem extrahendo

$$\frac{dp}{dt}=\sqrt{(\varDelta+2Bp+Cpp)}.$$

Cum igitur sit $\frac{dp}{dt}=\sqrt{X}+\sqrt{Y}$, aequatio integralis, quam sumus adepti, erit

$$\sqrt{X}+\sqrt{Y}=\sqrt{(\varDelta+2B(x+y)+C(x+y)^2)},$$

quae adeo est algebraica; ubi notetur esse

$$X=A+Bx+Cxx \quad \text{et} \quad Y=A+By+Cyy.$$

29. Sumamus igitur quadrata et nostra aequatio integralis erit

$$2A+B(x+y)+C(x^2+y^2)+2\sqrt{XY}=\varDelta+2B(x+y)+C(x+y)^2$$

sive

$$2A-B(x+y)-2Cxy+2\sqrt{XY}=\varDelta,$$

quae penitus ab irrationalitate liberata posito $\varDelta-2A=\Gamma$ praebebit

$$4AA+4AB(x+y)+4AC(xx+yy)+4BBxy+4BCxy(x+y)$$
$${}^{2}+2\Gamma B(x+y)+4\Gamma Cxy+BB(x+y)^2+4BCxy(x+y)+4CCxxyy$$

$$AA-\Gamma^2)+2B(2A-\Gamma)(x+y)+4(BB-\Gamma C)xy$$
$$+4AC(xx+yy)-B^2(x+y)^2=0.$$

30. Quodsi iam hanc aequationem rationalem cum formula *canonica*, qua olim sum usus ad huiusmodi integrationes expediendas, comparemus, quae erat

$$\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy=0,$$

dum scilicet loco $(x+y)^2$ scribamus $(xx+yy)+2xy$, reperiemus fore

$$\alpha=4AA-\Gamma^2,\quad \beta=B(2A-\Gamma),\quad \gamma=4AC-B^2,\quad \delta=BB-2\Gamma C.$$

31. Alio vero insuper modo eandem aequationem differentialem propositam integrare poterimus introducendo litteram $q=x-y$; tum enim habebimus

$$\frac{2\,ddq}{dt^2}=X'-Y'.$$

At vero erit

$$X'-Y'=2Cq+3Dq(x+y)+4Eq(xx+xy+yy),$$

ubi iterum patet statui debere tam $D=0$ quam $E=0$, ut integratio multiplicando per dq succedat. Hoc autem notato erit integrale

$$\frac{dq^2}{dt^2}=\text{Const.}+Cqq \quad \text{ideoque} \quad \frac{dq}{dt}=\sqrt{(\Delta+Cqq)}.$$

32. Cum igitur sit $\frac{dq}{dt}=\sqrt{X}-\sqrt{Y}$, hoc integrale ita erit expressum

$$\sqrt{X}-\sqrt{Y}=\sqrt{(\Delta+Cqq)},$$

quae aequatio sumtis quadratis abit in hanc

$$2A+B(x+y)+C(xx+yy)-2\sqrt{XY}=\Delta+C(x-y)^2$$

sive

$$2A+B(x+y)+2Cxy-2\sqrt{XY}=\Delta,$$

unde fit

$$2\sqrt{XY}=2A-\Delta+B(x+y)+2Cxy,$$

ubi si ponatur $2A-\Delta=-\Gamma$, aequatio ab ante inventa prorsus non discrepat.

33. Quodsi autem proposita fuisset aequatio

$$\frac{dx}{\sqrt{(A+Bx+Cxx)}}+\frac{dy}{\sqrt{(A+By+Cyy)}}=0,$$

integralia ante inventa ad hunc casum referentur, si modo loco $\sqrt{Y}$ scribatur $-\sqrt{Y}$; unde patet pro hoc casu haberi hanc aequationem

$$\sqrt{X}-\sqrt{Y}=\sqrt{\left(\Delta+2B(x+y)+C(x+y)^2\right)}$$

vel etiam

$$\sqrt{X}+\sqrt{Y}=\sqrt{\left(\Delta+C(x-y)^2\right)}.$$

34. Hic singularis casus occurrit, quando formulae $A+Bx+Cxx$ sunt quadrata. Sit enim

$$X=(a+bx)^2 \quad \text{et} \quad Y=(a+by)^2$$

ideoque

$$A=aa, \quad B=2ab, \quad C=bb;$$

tum enim prior forma integralis erit

$$b(x-y)=\sqrt{\left(\Delta+4ab(x+y)+bb(x+y)^2\right)}$$

sumtisque quadratis

$$-4bbxy=\Delta+4ab(x+y)$$

ideoque

$$\Delta=a(x+y)+bxy,$$

cuius aequationis differentiale est

$$a(dx+dy)+b(xdy+ydx)=0 \quad \text{ideoque} \quad dx(a+by)+dy(a+bx)=0.$$

Sin autem altera formula utatur, erit

$$2a+b(x+y)=\sqrt{\left(\Delta+bb(x-y)^2\right)},$$

unde quadratis sumtis positoque $\Delta-4aa=\Gamma$ prodit ut ante

$$\Gamma=a(x+y)+bxy.$$

ANALYSIS PRO INTEGRANDA AEQUATIONE

$$\frac{dx}{\sqrt{X}} = \frac{dy}{\sqrt{Y}}$$

EXISTENTE

$$X = A + Bx + Cxx + Dx^3 + Ex^4 \quad \text{ET} \quad Y = A + By + Cyy + Dy^3 + Ey^4$$

35. Introducto iterum elemento dt, ut sit

$$\frac{dx}{dt} = \sqrt{X} \quad \text{et} \quad \frac{dy}{dt} = \sqrt{Y}$$

ideoque sumtis quadratis

$$\frac{dx^2}{dt^2} = X \quad \text{et} \quad \frac{dy^2}{dt^2} = Y,$$

statuamus $x + y = p$ et $x - y = q$, et quia hinc prodit

$$dx^2 - dy^2 = dpdq,$$

erit

$$\frac{dpdq}{dt^2} = X - Y = B(x-y) + C(x^2 - y^2) + D(x^3 - y^3) + E(x^4 - y^4).$$

36. Quoniam igitur est $x = \frac{p+q}{2}$ et $y = \frac{p-q}{2}$, his valoribus introductis reperietur

$$X - Y = Bq + Cpq + \frac{1}{4}Dq(3pp + qq) + \frac{1}{2}Epq(pp + qq),$$

unde per q dividendo oritur

$$\frac{dpdq}{qdt^2} = B + Cp + \frac{1}{4}D(3pp + qq) + \frac{1}{2}Ep(pp + qq).$$

37. Nunc etiam formulas quadratas primo exhibitas differentiemus et statuendo ut ante

$$dX = X'dx \quad \text{et} \quad dY = Y'dy$$

habebimus

$$\frac{2ddx}{dt^2} = X' \quad \text{et} \quad \frac{2ddy}{dt^2} = Y'$$

hincque addendo

$$\frac{2ddp}{dt^2} = X' + Y'.$$

Cum vero sit

$$X' = B + 2Cx + 3Dxx + 4Ex^3 \quad \text{et} \quad Y' = B + 2Cy + 3Dyy + 4Ey^3,$$

erit

$$X' + Y' = 2B + 2Cp + \frac{3}{2}D(pp + qq) + Ep(pp + 3qq),$$

ita ut substituto hoc valore fiat

$$\frac{ddp}{dt^2} = B + Cp + \frac{3}{4}D(pp + qq) + \frac{1}{2}Ep(pp + 3qq),$$

a qua aequatione si priorem pro $\frac{dpdq}{qdt^2}$ subtrahamus, remanebit sequens

$$\frac{ddp}{dt^2} - \frac{dpdq}{qdt^2} = \frac{1}{2}Dqq + Epqq.$$

38. Haec iam aequatio per qq divisa producit istam

$$\frac{1}{dt^2}\left(\frac{ddp}{qq} - \frac{dpdq}{q^3}\right) = \frac{1}{2}D + Ep,$$

quae ducta in $2dp$ manifesto fit integrabilis; prodit enim

$$\frac{dp^2}{qqdt^2} = \varDelta + Dp + Epp,$$

ex qua radice extracta colligitur

$$\frac{dp}{qdt} = \sqrt{(\varDelta + Dp + Epp)}.$$

Cum igitur posuerimus $p = x + y$ et $q = x - y$, erit

$$\frac{dp}{dt} = \sqrt{X} + \sqrt{Y},$$

resultat haec aequatio integralis algebraica

$$\frac{\sqrt{X} + \sqrt{Y}}{x - y} = \sqrt{\left(\varDelta + D(x + y) + E(x + y)^2\right)},$$

est ipsa forma ab Illustri La Grange inventa.

39. Evolvamus ulterius hanc formam ac sumtis quadratis erit

$$\frac{X+Y+2\sqrt{XY}}{(x-y)^2} = \Delta + D(x+y) + E(x+y)^2.$$

Est vero

$$X+Y = 2A + B(x+y) + C(xx+yy) + D(x^3+y^3) + E(x^4+y^4);$$

unde si auferamus

$$(D(x+y) + E(x+y)^2)(x-y)^2,$$

remanebit

$$2A + B(x+y) + C(x^2+y^2) + Dxy(x+y) + 2Exxyy,$$

quo substituto aequatio integralis erit

$$\frac{2A + B(x+y) + C(x^2+y^2) + Dxy(x+y) + 2Ex^2y^2 + 2\sqrt{XY}}{(x-y)^2} = \Delta.$$

40. Haec aequatio aliquanto concinnior reddi potest subtrahendo utrinque C et statuendo $\Delta - C = \Gamma$; habebitur enim hoc facto

$$\frac{2A + B(x+y) + 2Cxy + Dxy(x+y) + 2Exxyy + 2\sqrt{XY}}{(x-y)^2} = \Gamma,$$

unde deducimus

$$2\sqrt{XY} = \Gamma(x-y)^2 - 2A - B(x+y) - 2Cxy - Dxy(x+y) - 2Exxyy,$$

sive ponendo

$$2A + B(x+y) + 2Cxy + Dxy(x+y) + 2Exxyy = V$$

aequatio nostra erit

$$2\sqrt{XY} = \Gamma(x-y)^2 - V,$$

quae sumtis quadratis abit in hanc

$$4XY = \Gamma^2(x-y)^4 - 2\Gamma V(x-y)^2 + VV$$

sive

$$4XY - VV = \Gamma^2(x-y)^4 - 2\Gamma V(x-y)^2.$$

41. Facta nunc substitutione erit

$$4XY = 4A^2 + 4AB(x+y) + 4AC(xx+yy) + 4AD(x^3+y^3) + 4AE(x^4+y^4)$$
$$+ 4BBxy + 4BCxy(x+y) + 4BDxy(xx+yy) + 4BExy(x^3+y^3)$$
$$+ 4CCxxyy + 4CDxxyy(x+y) + 4CExxyy(xx+yy)$$
$$+ 4DDx^3y^3 + 4DEx^3y^3(x+y) + 4EEx^4y^4.$$

At vero porro colligitur fore

$$VV = 4AA + 4AB(x+y) + 8ACxy + 4ADxy(x+y) + 8AExxyy$$
$$+ BB(x+y)^2 + 4BCxy(x+y) + 2BDxy(x+y)^2 + 4BE(x+y)xxyy$$
$$+ 4CCxxyy + 4CD(x+y)xxyy + 8CEx^3y^3$$
$$+ DDxxyy(x+y)^2 + 4DEx^3y^3(x+y) + 4EEx^4y^4.$$

42. Quodsi iam posteriorem formulam a priore subtrahamus et singulos terminos ordine analogos disponamus, reperiemus

$$4XY - VV = 4AC(x-y)^2 + 4AD(x+y)(x-y)^2$$
$$+ 4AE(x+y)^2(x-y)^2 - B^2(x-y)^2 + 2BDxy(x-y)^2 + 4BExy(x+y)(x-y)^2$$
$$+ 4CExxyy(x-y)^2 - DDxxyy(x-y)^2,$$

quae expressio factorem habet communem $(x-y)^2$, per quem ergo si dividamus, perveniemus ad hanc aequationem concinniorem

$$4AC + 4AD(x+y) + 4AE(x+y)^2$$
$$- BB + 2BDxy + 4BExy(x+y) + (4CE - DD)xxyy$$
$$= \Gamma\Gamma(x-y)^2 - 4\Gamma A - 2\Gamma B(x+y) - 4\Gamma Cxy - 2\Gamma Dxy(x+y) - 4\Gamma Exxyy.$$

43. Transferamus nunc omnes terminos ad partem sinistram et loco $(x+y)^2$ scribamus $(xx+yy)+2xy$, tum vero $(xx+yy)-2xy$ loco $(x-y)^2$, quo facto talis oritur aequatio meae canonicae respondens

$$0 = \left\{\begin{array}{llll} 4AC & +4AD(x+y) + 4AE(x^2+y^2) & +2BDxy + 4BExy(x+y) & +4CExxyy \\ -BB & +2\Gamma B(x+y) - \Gamma\Gamma\ (x^2+y^2) & +8AExy + 2\Gamma Dxy(x+y) & -DDxxyy \\ +4\Gamma A & & +2\Gamma^2 xy & +4\Gamma Exxyy \\ & & +4\Gamma Cxy & \end{array}\right.$$

44. Hinc ergo pro aequatione canonica litterae graecae α, β, γ, δ etc. per latinas A, B, C, D, E una cum constante Γ sequenti modo determinantur

$$\begin{aligned}
\alpha &= 4AC + 4\Gamma A - BB\\
\beta &= 2AD + \Gamma B\\
\gamma &= 4AE - \Gamma\Gamma\\
\delta &= BD + 4AE + \Gamma^2 + 2\Gamma C\\
\varepsilon &= 2BE + \Gamma D\\
\zeta &= 4CE + 4\Gamma E - DD,
\end{aligned}$$

ita ut aequatio canonica, qua olim sum usus, sit

$$\alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+y) + \zeta xxyy = 0.$$

45. Haec autem aequatio integralis ad rationalitatem perducta latius patet quam aequatio proposita differentialis $\frac{dx}{\sqrt{X}} - \frac{dy}{\sqrt{Y}} = 0$; simul enim complectitur integrale huius $\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0$. Scilicet haec aequatio complectitur duos factores, quorum alteruter alterutri satisfacit. Ex genesi autem patet hanc aequationem esse productum ex his factoribus

$$\Delta(x-y)^2 - V + 2\sqrt{XY} \quad \text{et} \quad \Delta(x-y)^2 - V - 2\sqrt{XY}.$$

46. Supra iam observavimus eiusdem aequationis differentialis integrale hoc quoque modo exhiberi posse

$$\frac{M\sqrt{Y} + N\sqrt{X}}{(x-y)^3} = C$$

(vide § 8 et praec.) existente

$$\begin{aligned}
M &= 4A + B(3x+y) + 2Cx(x+y) + Dxx(x+3y) + 4Ex^3y,\\
N &= 4A + B(3y+x) + 2Cy(x+y) + Dyy(y+3x) + 4Exy^3,
\end{aligned}$$

ubi notasse iuvabit esse

$$\begin{aligned}
M + N &= 8A + 4B(x+y) + 2C(x+y)^2 + D(x+y)^3 + 4Exy(xx+yy),\\
M - N &= 2B(x-y) + 2C(x+y)(x-y) + D(x-y)(x^2+4xy+y^2)\\
&\quad + 4Exy(x+y)(x-y).
\end{aligned}$$

Interim tamen haud facile intelligitur, quomodo haec forma cum ante inventa consentiat, dum tamen de consensu certi esse possumus.

47. Ex iis, quae hactenus sunt allata, satis liquet eandem aequationem integralem innumeris modis exhiberi posse, prout constans arbitraria alio atque alio modo repraesentatur; unde plurimum intererit certam legem stabilire, secundum quam quovis casu constantem illam arbitrariam exprimere velimus. Hunc in finem ista regula observetur, ut perpetuo integralia ita capiantur, ut posito $y = 0$ fiat $x = k$ hincque secundum legem compositionis $X = K$ existente

$$K = A + Bk + Ckk + Dk^3 + Ek^4.$$

Hac enim lege observata omnia integralia, utcunque diversa videantur, ad perfectum consensum perduci poterunt. Hoc igitur modo quae hactenus invenimus, sequentibus theorematibus complectamur.

THEOREMA 1

48. *Si haec aequatio differentialis*

$$\frac{dx}{a + bx + cxx} - \frac{dy}{a + by + cyy} = 0$$

ita integretur, ut posito $y = 0$ fiat $x = k$, integrale ita se habebit

$$\frac{2a + b(x + y) + 2cxy}{x - y} = \frac{2a + bk}{k}.$$

THEOREMA 2

49. *Si haec aequatio differentialis*

$$\frac{dx}{a + bx + cxx} + \frac{dy}{a + by + cyy} = 0$$

ita integretur, ut posito $y = 0$ fiat $x = k$, integrale supra triplici modo est inventum; erit enim

$$\text{I.}\quad \frac{b + c(x + y)}{cxy - a} = -\frac{b + ck}{a}.$$

$$\text{II.}\quad \frac{a(x + y) + bxy}{cxy - a} = -k$$

$$\text{III.}\quad \frac{b + c(x + y)}{a(x + y) + bxy} = \frac{b + ck}{ak}.$$

THEOREMA 3

50. *Si haec aequatio differentialis*

$$\frac{dx}{\sqrt{(A+Bx+Cxx)}} - \frac{dy}{\sqrt{(A+By+Cyy)}} = 0$$

ita integretur, ut posito $y = 0$ *fiat* $x = k$, *integrale erit*

$$-B(x+y) - 2Cxy + 2\sqrt{(A+Bx+Cxx)}\sqrt{(A+By+Cyy)}$$
$$= -Bk + 2\sqrt{A(A+Bk+Ckk)}$$

sive

$$B(k-x-y) - 2Cxy = 2\sqrt{A(A+Bk+Ckk)}$$
$$-2\sqrt{(A+Bx+Cxx)}\sqrt{(A+By+Cyy)}.$$

COROLLARIUM

51. Hinc ergo patet, si aequatio differentialis proposita fuerit ista

$$\frac{dx}{\sqrt{(A+Bx+Cxx)}} + \frac{dy}{\sqrt{(A+By+Cyy)}} = 0$$

eaque integretur ita, ut posito $y = 0$ fiat $x = k$, integrale fore

$$B(k-x-y) - 2Cxy$$
$$= 2\sqrt{(A+Bx+Cxx)}\sqrt{(A+By+Cyy)} - 2\sqrt{A(A+Bk+Ckk)}.$$

THEOREMA 4

52. *Si posito brevitatis gratia*

$$X = A + Bx + Cxx + Dx^3 + Ex^4,$$
$$Y = A + By + Cyy + Dy^3 + Ey^4,$$
$$K = A + Bk + Ckk + Dk^3 + Ek^4$$

haec proponetur aequatio differentialis

$$\frac{dx}{\sqrt{X}} - \frac{dy}{\sqrt{Y}} = 0,$$

quae ita integrari debeat, ut posito $y=0$ fiat $x=k$, eius integrale ita erit expressum

$$\frac{2A+B(x+y)+2Cxy+Dxy(x+y)+2Exxyy+2\sqrt{XY}}{(x-y)^2}=\frac{2A+Bk+2\sqrt{AK}}{kk}.$$

Sin autem aequatio proposita fuerit

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0,$$

eius integrale erit

$$\frac{2A+B(x+y)+2Cxy+Dxy(x+y)+2Exxyy-2\sqrt{XY}}{(x-y)^2}=\frac{2A+Bk-2\sqrt{AK}}{kk}.$$

COROLLARIUM 1

53. Quodsi hiç ponamus $D=0$ et $E=0$, casus oritur theorematis tertii pro aequatione

$$\frac{dx}{\sqrt{(A+Bx+Cxx)}}-\frac{dy}{\sqrt{(A+By+Cyy)}}=0,$$

cuius ergo integrale hinc erit

$$\frac{2A+B(x+y)+2Cxy+2\sqrt{(A+Bx+Cxx)(A+By+Cyy)}}{(x-y)^2}$$
$$=\frac{2A+Bk+2\sqrt{A(A+Bk+Ckk)}}{kk};$$

quae forma si cum superiori comparetur, formulae irrationales eliminari poterunt. Quoniam enim ex priore est

$$2\sqrt{XY}=2\sqrt{A(A+Bk+Ckk)}-B(k-x-y)+2Cxy,$$

erit hoc integrale postremum

$$\frac{2A+B(2x+2y-k)+4Cxy+2\sqrt{A(A+Bk+Ckk)}}{(x-y)^2}=\frac{2A+Bk+2\sqrt{A(A+Bk+Ckk)}}{kk},$$

unde statim deduci potest aequatio canonica

$$\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy=0.$$

COROLLARIUM 2

54. Ponamus nunc esse $A = 0$ et $B = 0$, ut sit

$X = xx(C + Dx + Exx)$ et $Y = yy(C + Dy + Eyy)$ et $K = kk(C + Dk + Ekk)$;

aequatio differentialis integranda fiet

$$\frac{dx}{x\sqrt{(C + Dx + Exx)}} - \frac{dy}{y\sqrt{(C + Dy + Eyy)}} = 0,$$

cuius ergo integrale erit

$$\frac{xy(2C + D(x + y) + 2Exy) + 2xy\sqrt{(C + Dx + Exx)(C + Dy + Eyy)}}{(x - y)^2} = \Delta,$$

atque hic constantem Δ per k definire non licebit; positio enim $y = 0$ incongruum iam involvit. Interim tamen et haec integratio maxime est memoratu digna.

COROLLARIUM 3

55. Quodsi autem in hac postrema integratione loco x et y scribamus $\frac{1}{x}$ et $\frac{1}{y}$, primo aequatio differentialis erit

$$\frac{dy}{\sqrt{(Cyy + Dy + E)}} - \frac{dx}{\sqrt{(Cxx + Dx + E)}} = 0;$$

tum vero integrale sequentem induet formam

$$\frac{2Cxy + D(x + y) + 2E + 2\sqrt{(Cxx + Dx + E)(Cyy + Dy + E)}}{(y - x)^2}$$
$$= \Delta = \frac{Dk + 2E + 2\sqrt{E(Ckk + Dk + E)}}{kk}.$$

Si igitur hic loco literarum E, D, C scribamus A, B, C, prodibit aequatio differentialis supra tractata

$$\frac{dx}{\sqrt{(A + Bx + Cxx)}} - \frac{dy}{\sqrt{(A + By + Cyy)}} = 0,$$

cuius ergo integrale erit

$$\frac{2A+B(x+y)+2Cxy+2\sqrt{(A+Bx+Cxx)(A+By+Cyy)}}{(x-y)^2}$$
$$=\frac{Bk+2A+2\sqrt{A(A+Bk+Ckk)}}{kk},$$

quae egregie convenit cum ea in coroll. 1 data.

COROLLARIUM 4

56. Contemplemur nunc etiam casum, quo formula

$$A+Bx+Cxx+Dx^3+Ex^4$$

fit quadratum, quod sit $(a+bx+cxx)^2$, ita ut iam habeamus

$$A=aa,\quad B=2ab,\quad C=bb+2ac,\quad D=2bc,\quad E=cc,$$

tum vero

$$\sqrt{X}=a+bx+cxx,\quad \sqrt{Y}=a+by+cyy,\quad \sqrt{K}=a+bk+ckk$$

atque aequatio differentialis pro priore casu erit

$$\frac{dx}{a+bx+cxx}-\frac{dy}{a+by+cyy}=0,$$

cuius ergo integrale erit

$$\left\{\begin{matrix}2aa+2ab(x+y)+2(bb+2ac)xy+2bcxy(x+y)\\ +2ccxxyy+2(a+bx+cxx)(a+by+cyy)\end{matrix}\right\}:(x-y)^2=\varDelta,$$

quae reducitur ad

$$\frac{aa+ab(x+y)+(bb+2ac)xy+bcxy(x+y)+ccxxyy}{(x-y)^2}=\frac{aa+abk}{kk}.$$

Quodsi iam utrinque addamus $\frac{1}{4}bb$, prodibit

$$\frac{\left(a+\frac{1}{2}b(x+y)+cxy\right)^2}{(x-y)^2}=\frac{\left(a+\frac{1}{2}bk\right)^2}{k^2},$$

unde extracta radice obtinetur forma integralis in theoremate primo assignata.

57. Sin autem hoc modo alterum casum aequationis

$$\frac{dx}{a+bx+cxx} + \frac{dy}{a+by+cyy} = 0$$

evolvere volimus, pervenimus ad hanc aequationem

$$\frac{2aa+2ab(x+y)+2(bb+2ac)xy+2bcxy(x+y)+2ccxxyy}{(x-y)^2}$$
$$-\frac{2(a+bx+cxx)(a+by+cyy)}{(x-y)^2} = \Delta,$$

quae evoluta praebet $\Delta = -2ac$, haecque aequatio manifesto est absurda et nihil circa integrale quaesitum declarat, cuius rationem maximi momenti erit perscrutari.

INSIGNE PARADOXON

58. *Cum huius aequationis differentialis*

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0$$

integrale in genere inventum sit

$$\frac{2A+B(x+y)+2Cxy+Dxy(x+y)+2Exxyy-2\sqrt{XY}}{(x-y)^2} = \Delta,$$

casu autem, quo statuitur

$$\sqrt{X} = a+bx+cxx \quad et \quad \sqrt{Y} = a+by+cyy,$$

aequatio absurda inde oriatur, quaeritur enodatio huius insignis difficultatis ac praecipue modus hinc verum integralis valorem investigandi.

ENODATIO PARADOXI

59. Quemadmodum scilicet in Analysi eiusmodi formulae occurrere solent, quae certis casibus indeterminatae atque adeo nihil plane significare videntur, ita hic simile quid usu venit, sed longe alio modo, cum neque ad fractionem, cuius numerator et denominator simul evanescunt, neque ad differentiam inter duo infinita perveniatur, quod exemplum eo magis est notatu dignum, quod non memini similem casum mihi umquam se obtulisse. Istud singulare phaenomenon se nimirum exerit, quando ambae formulae X et Y evadunt qua-

drata, ad quod ergo resolvendum ad simile artificium recurri oportet, quo formulae X et Y non ipsis quadratis aequales, sed ab iis infinite parum discrepare assumuntur.

60. Statuamus igitur

$$X=(a+bx+cxx)^2+\alpha \quad \text{et} \quad Y=(a+by+cyy)^2+\alpha,$$

ita ut pro litteris maiusculis A, B, C, D, E fiat $A=aa+\alpha$, $B=2ab$, $C=2ac+bb$, $D=2bc$, $E=cc$, ubi α denotat quantitatem infinite parvam deinceps nihilo aequalem ponendam. Hinc ergo si brevitatis gratia ponamus

$$a+bx+cxx=R \quad \text{et} \quad a+by+cyy=S,$$

erit

$$\sqrt{X}=R+\frac{\alpha}{2R} \quad \text{et} \quad \sqrt{Y}=S+\frac{\alpha}{2S}.$$

61. Nunc igitur consideremus formam integralis primo inventam, quae erat

$$\frac{\sqrt{X}-\sqrt{Y}}{x-y}=\sqrt{\left(\varDelta+D(x+y)+E(x+y)^2\right)},$$

pro qua igitur habebimus

$$\sqrt{X}-\sqrt{Y}=R-S-\frac{\alpha(R-S)}{2RS}.$$

Quia vero hic erit $R-S=b(x-y)+c(xx-yy)$, fiet

$$\frac{R-S}{x-y}=b+c(x+y).$$

At posito brevitatis gratia $x+y=p$ erit

$$\frac{R-S}{x-y}=b+cp,$$

unde aequatio nostra erit

$$b+cp-\frac{\alpha(b+cp)}{2RS}=\sqrt{(\varDelta+2bcp+ccpp)}.$$

62. Sumantur nunc utrinque quadrata et aequatio nostra sequentem induet formam $bb-\frac{\alpha}{RS}(b+cp)^2=\varDelta$. Alteriores scilicet potestates ipsius α hic ubique praetermittuntur. Hic ergo ratio nostri paradoxi manifesto in oculos

incidit, quia posito $\alpha = 0$ oritur $bb = \Delta$; unde, ut Δ maneat constans arbitraria, evidens est differentiam inter bb et Δ etiam infinite parvam statui debere; quamobrem ponamus $\Delta = bb - \alpha\Gamma$ ac obtinebitur ista aequatio penitus determinata $\frac{(b+cp)^2}{RS} = \Gamma$ sive

$$(b + c(x + y))^2 = \Gamma(a + bx + cxx)(a + by + cyy),$$

quae forma non multum discrepat a formula supra inventa.

63. Haec quidem forma magis est complicata quam solutiones § 24 et seqq. inventae, sequenti autem artificio ad formam simplicissimam redigi poterit. Cum haec fractio $\frac{RS}{(b+cp)^2}$ debeat esse quantitas constans, sit ea $= F$, ut esse debeat $F(cp + b)^2 = RS$, et quemadmodum hic posuimus $x + y = p$, ponamus porro $xy = u$ fietque

$$RS = aa + abp + ac(pp - 2u) + bbu + bcpu + ccuu$$

atque aequatio iam secundum potestates ipsius p disposita erit

$$F(cp + b)^2 = acpp + abp + bcpu + bbu + aa - 2acu + ccuu;$$

ubi primo utrinque dividamus, quatenus fieri potest, per $cp + b$, ac reperietur

$$F(cp + b) = ap + bu + \frac{(a - cu)^2}{cp + b}.$$

Dividamus nunc porro per $cp + b$, quatenus fieri potest, ac fiet

$$F = \frac{a}{c} - \frac{b}{c} \cdot \frac{a - cu}{cp + b} + \frac{(a - cu)^2}{(cp + b)^2}.$$

64. Hac forma inventa si statuamus

$$\frac{a - cu}{cp + b} = V,$$

erit

$$F = \frac{a}{c} - \frac{b}{c} V + VV.$$

Cum igitur ista expressio aequari debeat quantitati constanti, evidens est

ipsam quantitatem V constantem esse debere, ita ut iam nostrum integrale reductum sit ad hanc formam

$$\frac{a-cu}{cp+b}=\frac{a-cxy}{c(x+y)+b}=\text{Const.}$$

Subtrahamus utrinque $\frac{a}{b}$ fietque

$$\frac{cxy+a(x+y)}{b+c(x+y)}=\text{Const.},$$

quae forma per priorem divisa producit hanc

$$\frac{a(x+y)+cxy}{cxy-a}=\text{Const.},$$

quae formae conveniunt cum supra exhibitis.

THEOREMA 5

65. *Si in genere haec ratio designandi adhibeatur, ut sit*

$$Z=A+Bz+Czz+Dz^3+Ez^4,$$

atque valor huius formulae integralis $\int\frac{dz}{\sqrt{Z}}$ *ita sumtus, ut evanescat posito* $z=0$, *designetur hoc charactere* $\Pi:z$, *tum, ut fiat* $\Pi:k=\Pi:x\pm\Pi:y$, *necesse est, ut inter quantitates* k, x, y *ista relatio subsistat*

$$\frac{2A+B(x+y)+2Cxy+Dxy(x+y)+2Exxyy\mp2\sqrt{XY}}{(x-y)^2}=\frac{2A+Bk\mp2\sqrt{AK}}{kk},$$

cuius ratio ex superioribus est manifesta.

Cum enim k denotet quantitatem constantem, erit

$$d.\,\Pi:x\pm d.\,\Pi:y=0\quad\text{sive}\quad\frac{dx}{\sqrt{X}}\pm\frac{dy}{\sqrt{Y}}=0,$$

cuius integrale modo ante vidimus ita exprimi

$$\frac{2A+B(x+y)+2Cxy+Dxy(x+y)+2Exxyy\mp2\sqrt{XY}}{(x-y)^2}=\Delta.$$

Quare cum esse debeat $\Pi:x + \Pi:y = \Pi:k$, manifestum est posito $y = 0$ fieri debere $\Pi:x = \Pi:k$ ideoque $x = k$, unde constans indefinita A eodem prorsus modo definitur, uti est exhibita.

COROLLARIUM 1

66. Hinc si formula $\Pi:z$ exprimat arcum cuiuspiam lineae curvae abscissae sive applicatae Z respondentem, in hac curva omnes arcus eodem modo inter se comparare licebit, quo arcus circulares inter se comparantur, quandoquidem propositis duobus arcubus $\Pi:x$ et $\Pi:y$ tertius arcus $\Pi:k$ semper exhiberi poterit vel summae vel differentiae eorum arcuum aequalis.

COROLLARIUM 2

67. Ita si in hac forma $\Pi:k = \Pi:x + \Pi:y$ statuatur $y = x$, prodibit $\Pi:k = 2\Pi:x$ sicque arcus reperitur duplo alterius aequalis. At vero si in nostra forma faciamus $y = x$, tam numerator quam denominator in nihilum abeunt. Ut autem eius verum valorem eruamus, utamur aequatione primum § 38 inventa

$$\frac{\sqrt{X} - \sqrt{Y}}{x - y} = \sqrt{(A + D(x+y) + E(x+y)^2)}$$

et iam in membro sinistro spectetur y ut constans; ipsi x vero valorem tribuamus infinite parum discrepantem sive, quod eodem redit, loco numeratoris et denominatoris eorum differentialia substituantur sumta sola x variabili hocque modo pro casu $y = x$ membrum sinistrum evadit $\frac{X'}{2\sqrt{X}}$, ubi est $X' = B + 2Cx + 3Dxx + 4Ex^3$. Nunc ergo sumtis quadratis habebitur

$$\frac{X'X'}{4X} = A + 2Dx + 4Exx$$

existente A ut ante $= \frac{2A + Bk}{kk} + 2\sqrt{AK}$.

COROLLARIUM 3

68. Verum sine his ambagibus duplicatio arcus ex altera forma

$$\Pi:k = \Pi:x - \Pi:y$$

deduci potest ponendo $y = -k$, siquidem hinc fit $\Pi:x = 2\Pi:k$, pro quo ergo

casu relatio inter x et k hac aequatione exprimetur

$$\frac{2A+B(k+x)+2Ckx+Dkx(k+x)+2Ekkxx+2\sqrt{KX}}{(x-k)^2}=\frac{2A+Bk+2\sqrt{AK}}{kk}.$$

Facile autem patet, quomodo hinc ad triplicationem, quadruplicationem et quamlibet multiplicationem arcuum progredi debeat, quod argumentum olim fusius sum tractatus.

THEOREMA 6

69. *Si in formis supra inventis ponatur tam* $B=0$ *quam* $D=0$, *ut sit*

$$X=A+Cxx+Ex^4 \quad et \quad Y=A+Cyy+Ey^4 \quad et \quad K=A+Ckk+Ek^4,$$

tum si ista aequatio $\frac{dx}{\sqrt{X}}\pm\frac{dy}{\sqrt{Y}}=0$ *ita integretur, ut posito* $y=0$ *fiat* $x=k$, *tum aequatio integralis erit*

$$\frac{A+Cxy+Exxyy\mp\sqrt{XY}}{(x-y)^2}=\frac{A\mp\sqrt{AK}}{kk}.$$

COROLLARIUM 1

70. Hic notari meretur istum casum adhuc alio modo ex forma generali deduci posse, si scilicet sumatur $A=0$ et $E=0$, tum enim prodit ista aequatio differentialis

$$\frac{dx}{\sqrt{(Bx+Cxx+Dx^3)}}\pm\frac{dy}{\sqrt{(By+Cyy+Dy^3)}}=0,$$

cuius ergo integrale erit

$$\frac{B(x+y)+2Cxy+Dxy(x+y)\mp2\sqrt{(Bx+Cxx+Dx^3)(By+Cyy+Dy^3)}}{(x-y)^2}=\frac{Bk}{kk}=\frac{B}{k},$$

ubi valor constantis admodum simplex evasit. Nunc in his formulis loco x et y scribamus xx et yy, at vero loco litterarum B et D scribamus A et E fietque aequatio differentialis

$$\frac{dx}{\sqrt{(A+Cxx+Ex^4)}}\pm\frac{dy}{\sqrt{(A+Cyy+Ey^4)}}=0$$

cuius ergo integrale etiam hoc modo exprimetur

$$\frac{A(xx+yy)+2Cxxyy+Exxyy(xx+yy)\mp 2xy\sqrt{XY}}{(xx-yy)^2}=\frac{A}{kk}.$$

COROLLARIUM 2

71. Ecce ergo hac ratione pervenimus ad aliam integralis formam non minus notabilem priore atque adeo nunc ex earum combinatione formula radicalis $\sqrt{XY}$ eliminari poterit, quandoquidem ex posteriore fit

$$\mp 2\sqrt{XY}=\frac{A(xx-yy)^2}{kkxy}-\frac{A(xx+yy)}{xy}-2Cxy-Exy(xx+yy),$$

qui valor in priore substitutus conducit ad hanc aequationem rationalem

$$2A+2Cxy+2Exxyy+\frac{A(xx-yy)^2}{kkxy}-\frac{A(xx+yy)}{xy}-2Cxy-Exy(xx+yy)$$
$$=\frac{2A(x-y)^2}{kk}\mp\frac{2(x-y)^2\sqrt{AK}}{kk},$$

quae porro reducta et per $(x-y)^2$ divisa revocatur ad hanc formam

$$\frac{2A\mp 2\sqrt{AK}}{kk}=\frac{A(x+y)^2}{kkxy}-Exy-\frac{A}{xy}$$

sive ad hanc

$$\frac{A}{kk}(xx+yy-kk)-Exxyy\pm\frac{2xy\sqrt{AK}}{kk}=0,$$

quae egregie convenit cum aequatione canonica, qua olim sum usus, scilicet

$$0=\alpha+\gamma(xx+yy)+2\delta xy+\zeta xxyy,$$

si quidem est

$$\alpha=-A,\quad \gamma=+\frac{A}{kk},\quad 2\delta=\pm\frac{2\sqrt{AK}}{kk},\quad \zeta=-E.$$

COROLLARIUM 3

72. Methodo posteriore, qua hic usi sumus ad hanc aequationem integrandam, aequatio multo generalior tractari poterit, ubi in formulis radicalibus potestates usque ad sextam dimensionem assurgunt. Namque si tantum

statuamus $A = 0$, ut sit aequatio

$$\frac{dx}{\sqrt{x(B + Cx + Dxx + Ex^3)}} \pm \frac{dy}{\sqrt{y(B + Cy + Dyy + Ey^3)}} = 0,$$

eius integrale est

$$\frac{B(x+y) + 2Cxy + Dxy(x+y) + 2Exxyy}{(x-y)^2}$$
$$\mp \frac{2\sqrt{xy(B + Cx + Dxx + Ex^3)(B + Cy + Dyy + Ey^3)}}{(x-y)^2} = \frac{B}{k}.$$

Quod si iam hic loco x et y scribamus xx et yy, aequatio differentialis fiet

$$\frac{dx}{\sqrt{(B + Cxx + Dx^4 + Ex^6)}} \pm \frac{dy}{\sqrt{(B + Cyy + Dy^4 + Ey^6)}} = 0,$$

cuius ergo integrale erit

$$\frac{B(xx+yy) + 2Cxxyy + Dxxyy(xx+yy) + 2Ex^4y^4}{(xx-yy)^2}$$
$$\mp \frac{2xy\sqrt{(B + Cxx + Dx^4 + Ex^6)(B + Cyy + Dy^4 + Ey^6)}}{(xx-yy)^2} = \frac{B}{kk}.$$

Nunc autem ostendamus, quomodo ope methodi Illustris DE LA GRANGE idem integrale impetrari queat.

ANALYSIS
PRO INTEGRATIONE AEQUATIONIS DIFFERENTIALIS

$$\frac{dx}{\sqrt{X}} \pm \frac{dy}{\sqrt{Y}} = 0$$

EXISTENTE

$$X = B + Cxx + Dx^4 + Ex^6 \quad \text{ET} \quad Y = B + Cyy + Dy^4 + Ey^6$$

73. Posito igitur $\frac{dx}{\sqrt{X}} = dt$ erit $\frac{dy}{\sqrt{Y}} = \mp dt$ hincque sumtis quadratis

$$\frac{dx^2}{dt^2} = X \quad \text{et} \quad \frac{dy^2}{dt^2} = Y.$$

Hinc formentur hae aequationes

$$\frac{xxdx^2}{dt^2} = xxX \quad \text{et} \quad \frac{yydy^2}{dt^2} = yyY.$$

Iam introducantur duae novae variabiles p et q, ut sit

$$xx+yy=2p \quad \text{et} \quad xx-yy=2q,$$

ex quo fit

$$xdx+ydy=dp, \quad xdx-ydy=dq \quad \text{hincque} \quad xxdx^2-yydy^2=dpdq;$$

quamobrem habebimus

$$\frac{dpdq}{dt^2}=xxX-yyY,$$

quae aequatio dividatur per $xx-yy=2q$, prodibitque

$$\frac{dpdq}{2qdt^2}=\frac{xxX-yyY}{xx-yy},$$

quae forma valoribus pro X et Y substitutis dabit

$$\frac{dpdq}{2qdt^2}=B+2Cp+D(3pp+qq)+4E(p^3+pqq).$$

74. Nunc porro aequationes pro $\frac{dx^2}{dt^2}$ et $\frac{dy^2}{dt^2}$ differentiatae dabunt

$$\frac{2ddx}{dt^2}=X' \quad \text{et} \quad \frac{2ddy}{dt^2}=Y'.$$

Ex priore fit $\frac{2xddx}{dt^2}=xX'$, cui addatur $\frac{2dx^2}{dt^2}=2X$, ut prodeat

$$\frac{2(xddx+dx^2)}{dt^2}=\frac{2d.xdx}{dt^2}=xX'+2X.$$

Simili modo erit

$$\frac{2d.ydy}{dt^2}=yY'+2Y,$$

quae duae aequationes invicem additae dabunt

$$\frac{2d.dp}{dt^2}=\frac{2ddp}{dt^2}=xX'+yY'+2(X+Y).$$

Substitutis autem valoribus et facta substitutione respectu litterarum p et q reperitur

$$2X+2Y=4B+4Cp+4D(pp+qq)+4Ep(pp+3qq).$$

Deinde ob

$$xX' = 2Cxx + 4Dx^4 + 6Ex^6 \quad \text{et} \quad yY' = 2Cyy + 4Dy^4 + 6Ey^6$$

erit

$$xX' + yY' = 4Cp + 8D(pp + qq) + 12Ep(pp + 3qq),$$

ex quibus coniunctis fit

$$\frac{2ddp}{dt^2} = 4B + 8Cp + 12D(pp + qq) + 16Ep(pp + 3qq).$$

75. Ab hac formula subtrahatur supra inventa $\frac{dpdq}{2qdt^2}$ quater sumta ac remanebit

$$\frac{2ddp}{dt^2} - \frac{2dpdq}{qdt^2} = 8Dqq + 32Epqq.$$

Nunc utrinque multiplicetur per $\frac{dp}{qq}$ et prodibit

$$\frac{1}{dt^2}\left(\frac{2dpddp}{qq} - \frac{2dp^2dq}{q^3}\right) = 8Ddp + 32Epdp,$$

cuius integrale sponte se offert ita expressum

$$\frac{dp^2}{qqdt^2} = 4\varDelta + 8Dp + 16Epp$$

ideoque extracta radice

$$\frac{dp}{qdt} = 2\sqrt{(\varDelta + 2Dp + 4Epp)}.$$

76. Cum nunc sit

$$\frac{dp}{dt} = x\sqrt{X} \mp y\sqrt{Y} \quad \text{et} \quad 2q = xx - yy,$$

facta substitutione orietur haec aequatio

$$\frac{x\sqrt{X} \mp y\sqrt{Y}}{xx - yy} = \sqrt{\left(\varDelta + D(xx + yy) + E(xx + yy)^2\right)},$$

quae sumtis quadratis reducetur ad istam formam

$$\frac{xxX + yyY \mp 2xy\sqrt{XY}}{(xx - yy)^2} = \varDelta + D(xx + yy) + E(xx + yy)^2.$$

Est vero

$$xxX + yyY = B(xx + yy) + C(x^4 + y^4) + D(x^6 + y^6) + E(x^8 + y^8)$$

hincque pervenietur ad hanc aequationem

$$\frac{B(xx+yy) + C(x^4+y^4) + Dxxyy(xx+yy) + 2Ex^4y^4 \mp 2xy\sqrt{XY}}{(xx-yy)^2} = \varDelta.$$

77. Sumamus nunc ut supra constantem $\varDelta$ ita, ut posito $y = 0$ fiat

$$x = k \quad \text{et} \quad X = K = B + Ckk + Dk^4 + Ek^6,$$

et aequatio integralis induet hanc formam

$$\frac{B(xx+yy) + C(x^4+y^4) + Dxxyy(xx+yy) + 2Ex^4y^4 \mp 2xy\sqrt{XY}}{(xx-yy)^2} = \frac{B + Ckk}{kk},$$

quae aliquanto simplicior evadit, si utrinque subtrahamus C; erit enim

$$\frac{B(xx+yy) + 2Cxxyy + Dxxyy(xx+yy) + 2Ex^4y^4 \mp 2xy\sqrt{XY}}{(x^2-y^2)^2} = \frac{B}{kk},$$

quae egregie convenit cum integrali supra § 72 exhibito.

78. Hic casus notatu dignus se offert, dum $B = 0$; tum autem aequatio differentialis ita se habebit

$$\frac{dx}{x\sqrt{(C + Dxx + Ex^4)}} \pm \frac{dy}{y\sqrt{(C + Dyy + Ey^4)}} = 0,$$

cuius ergo integrale per constantem $\varDelta$ expressum erit

$$\frac{C(x^4+y^4) + Dxxyy(xx+yy) + 2Ex^4y^4 \mp 2xy\sqrt{XY}}{(xx-yy)^2} = \varDelta.$$

Hoc autem casu integratio non ita determinari potest, ut posito $y = 0$ fiat $x = k$, quia integrale posterioris membri hoc casu manifesto abit in infinitum; quamobrem alio modo integrationem determinari conveniet, veluti ut posito $y = b$ fiat $x = a$; tum autem erit ista constans

$$\varDelta = \frac{C(a^4+b^4) + Da^2b^2(aa+bb) + 2Ea^4b^4 \mp 2ab\sqrt{AB}}{(aa-bb)^2}$$

existente

$$A = C + Daa + Ea^4 \quad \text{et} \quad B = C + Dbb + Eb^4.$$

CONCLUSIO

79. Qui processum Analyseos hic usitatae comparare voluerit cum methodo, qua Illustris D. DE LA GRANGE usus est in Miscellan. Taur. Tom. IV, facile perspiciet eam multo facilius ad scopum desideratum perducere atque multo commodius ad quosvis casus applicari posse. Introduxerat autem vir illustrissimus in calculum formulam $\frac{dt}{T}$, cuius loco hic simplici elemento dt sumus usi, ac deinceps quantitatem T tanquam functionem litterarum p et q spectavit, quae positio satis difficiles calculos postulavit, dum nostra methodo longe concinnius easdem integrationes investigare licuit. Quanquam autem nullum est dubium, quin ista Analyseos species insigne incrementum polliceatur, tamen nondum patet, quemadmodum ad alias integrationes ea accommodari queat praeter hos ipsos casus, quos hic tractavimus et quos olim ex aequatione canonica derivaveram.

PLENIOR EXPLICATIO CIRCA COMPARATIONEM QUANTITATUM IN FORMULA INTEGRALI $\int\frac{Zdz}{\sqrt{(1+mzz+nz^4)}}$ CONTENTARUM DENOTANTE Z FUNCTIONEM QUAMCUNQUE RATIONALEM IPSIUS zz

Commentatio 581 indicis ENESTROEMIANI
Acta academiae scientiarum Petropolitanae 1781: II (1785), p. 3—22

1. Etsi hoc argumentum iam saepius tractavi atque Illustrissimus LA GRANGE plures egregias observationes super huiusmodi formulis cum publico communicavit, id tamen neutiquam adhuc satis exploratum, multo minus exhaustum est censendum, sed plurima adhuc maxime abscondita involvere videtur, quae profundissimam indagationem requirunt atque insignia incrementa Analyseos pollicentur. Imprimis autem ipsae operationes analyticae, quae me primum ad hanc investigationem perduxerunt, ita sunt comparatae, ut non nisi per plures ambages totum negotium conficiant, unde merito etiamnunc methodus directa ad easdem comparationes perducens maxime est desideranda. Praeterea vero universa haec investigatio multo latius patet quam ad eas formulas integrales, quas primo sum contemplatus, ubi pro littera Z tantum vel quantitatem constantem vel functionem integram ipsius zz huius formae $F+Gzz+Hz^4+Iz^6+Kz^8+$ etc. assumsi, quibus casibus ostendi propositis duabus quibuscunque quantitatibus huius generis semper tertiam eiusdem generis inveniri posse, quae a summa illarum discrepet quantitate algebraica, quae quidem evanescat casu, quo Z est tantum quantitas constans.

2. Nunc autem observavi easdem comparationes institui posse, si pro Z accipiatur functio quaecunque rationalis ipsius zz, quae scilicet habeat huiusmodi formam

$$\frac{F+Gzz+Hz^4+Iz^6+Kz^8+\text{etc.}}{\mathfrak{F}+\mathfrak{G}zz+\mathfrak{H}z^4+\mathfrak{I}z^6+\mathfrak{K}z^8+\text{etc.}},$$

ubi quidem hoc discrimen occurrit, quod differentia inter summam duarum huiusmodi formularum et tertiam formulam eiusdem generis inveniendam non amplius sit quantitas algebraica, veruntamen per logarithmos et arcus circulares semper exhiberi possit, ita ut nunc ista investigatio multo latius pateat, quam eam adhuc eram complexus. Atque hinc fortasse, si omnes operationes, quae ad hunc scopum manuducunt, debita attentione perpendantur, faciliorem viam aperire poterunt ad methodum directam perveniendi totumque hoc argumentum maxime abstrusum feliciori successu perscrutandi.

3. Quo autem haec omnia clarius perspici queant, denotet iste character $\Pi:z$ eam quantitatem transcendentem, quae ex integratione formulae propositae

$$\int\frac{Zdz}{\sqrt{(1+mzz+nz^4)}}$$

nascitur, dum integrale ita capi assumitur, ut evanescat posito $z=0$; unde statim manifestum est fore quoque $\Pi:0=0$. Deinde cum Z involvat tantum pares potestates ipsius z, cuiusmodi etiam in formula radicali insunt, evidens est, si loco z scribatur $-z$, tum valorem quoque istius formulae integralis ideoque etiam characteris $\Pi:z$ in sui negativum abire, ita ut sit $\Pi:(-z)=-\Pi:z$. His praenotatis si proponantur duae quaecunque huiusmodi quantitates $\Pi:p$ et $\Pi:q$, semper invenire licet tertiam quantitatem eiusdem generis $\Pi:r$, quae a summa illarum formularum $\Pi:p+\Pi:q$ differat quantitate vel algebraica vel saltem per logarithmos et arcus circulares assignabili. Regula vero, qua ex datis litteris p et q tertia r elicitur, semper manet eadem, quaecunque functio per litteram Z designetur; semper enim erit

$$r=\frac{p\sqrt{(1+mqq+nq^4)}+q\sqrt{(1+mpp+np^4)}}{1-nppqq}.$$

Hinc autem pro sequentibus combinationibus observasse iuvabit fore

$$\sqrt{(1+mrr+nr^4)}$$
$$=\frac{(mpq+\sqrt{(1+mpp+np^4)}\sqrt{(1+mqq+nq^4)})(1+nppqq)+2npq(pp+qq)}{(1-nppqq)^2}.$$

4. Non solum autem haec investigatio adstringitur ad huiusmodi formulas $\Pi : p$ et $\Pi : q$ pro arbitrio accipiendas, sed adeo ad quotcunque formulas datas potest extendi, ita ut, quotcunque huiusmodi formulae fuerint propositae, scilicet

$$\Pi : f + \Pi : g + \Pi : h + \Pi : i + \Pi : k + \text{ etc.},$$

semper nova huiusmodi formula $\Pi : r$ assignari possit, quae ab illarum summa discrepet quantitate vel algebraica vel saltem per logarithmos et arcus circulares assignabili. Quin etiam formulas illas, quas tanquam datas spectavimus, ita definire licebit, ut discrimen illud sive algebraicum sive a logarithmis arcubusque circularibus pendens prorsus evanescat, ita ut futurum sit

$$\Pi : r = \Pi : f + \Pi : g + \Pi : h + \Pi : i + \Pi : k + \text{ etc.}$$

Atque haec fere sunt, ad quae hanc investigationem generaliorem, quam hic exponere constitui, mihi quidem extendere licuit; quamobrem singulas operationes, quae me huc perduxerunt, succincte sum propositurus.

OPERATIO 1

5. Universam hanc investigationem inchoavi a consideratione huius aequationis algebraicae

$$\alpha + \gamma(xx + yy) + 2\delta xy + \zeta xxyy = 0,$$

ex qua, cum sit quadratica, tam pro x quam pro y radicem extrahendo colligitur vel

$$y = \frac{-\delta x + \sqrt{(-\alpha\gamma + (\delta\delta - \gamma\gamma - \alpha\zeta)xx - \gamma\zeta x^4)}}{\gamma + \zeta xx}$$

vel

$$x = \frac{-\delta y + \sqrt{(-\alpha\gamma + (\delta\delta - \gamma\gamma - \alpha\zeta)yy - \gamma\zeta y^4)}}{\gamma + \zeta yy},$$

ita ut hinc fiat

$$\sqrt{(-\alpha\gamma + (\delta\delta - \gamma\gamma - \alpha\zeta)xx - \gamma\zeta x^4)} = \gamma y + \delta x + \zeta xxy$$

et

$$\sqrt{(-\alpha\gamma + (\delta\delta - \gamma\gamma - \alpha\zeta)yy - \gamma\zeta y^4)} = \gamma x + \delta y + \zeta xyy.$$

6. Nunc litteras α, γ, δ, ζ ita definio, ut ambae formulae radicales ad formam

$$\sqrt{(1+mxx+nx^4)} \quad \text{et} \quad \sqrt{(1+myy+ny^4)}$$

reducantur, quem in finem facio

$$1.\ -\alpha\gamma = k, \quad 2.\ \delta\delta - \gamma\gamma - \alpha\zeta = mk \quad \text{et} \quad 3.\ -\gamma\zeta = nk;$$

ex quarum aequalitatum prima fit $\alpha = -\frac{k}{\gamma}$, ex tertia $\zeta = \frac{-nk}{\gamma}$, qui valores in secunda substituti praebent

$$\delta\delta = \gamma\gamma + \frac{nkk}{\gamma\gamma} + mk$$

ideoque

$$\delta = \sqrt{\left(\gamma\gamma + \frac{nkk}{\gamma\gamma} + mk\right)} = \frac{1}{\gamma}\sqrt{(\gamma^4 + m\gamma\gamma k + nkk)};$$

unde aequatio nostra nunc erit

$$-k + \gamma\gamma(xx+yy) + 2xy\sqrt{(\gamma^4 + m\gamma\gamma k + nkk)} - nkxxyy = 0;$$

hinc igitur ambae nostrae formulae irrationales erunt

$$\sqrt{k(1+mxx+nx^4)} = \gamma y + \frac{1}{\gamma}x\sqrt{(\gamma^4 + m\gamma\gamma k + nkk)} - \frac{nk}{\gamma}xxy,$$

$$\sqrt{k(1+myy+ny^4)} = \gamma x + \frac{1}{\gamma}y\sqrt{(\gamma^4 + m\gamma\gamma k + nkk)} - \frac{nk}{\gamma}xyy.$$

7. Cum nunc ambae quantitates x et y ita a se invicem pendeant, quemadmodum aequatio assumta declarat, litteras adhuc indefinitas γ et k ita definiamus, ut posito $x=0$ fiat $y=a$. Oportebit igitur esse $-k+\gamma\gamma aa=0$ ideoque $k=\gamma\gamma aa$, quo valore substituto aequatio nostra erit

$$0 = \gamma\gamma(xx+yy-aa) + 2\gamma\gamma xy\sqrt{(1+maa+na^4)} - n\gamma\gamma aaxxyy,$$

hincque fiet per $\gamma\gamma$ dividendo

$$0 = (xx+yy-aa) + 2xy\sqrt{(1+maa+na^4)} - naaxxyy.$$

Tum vero ambae nostrae formulae radicales ita exprimentur

$$\sqrt{(1+mxx+nx^4)} = \frac{y}{a} + \frac{x}{a}\sqrt{(1+maa+na^4)} - naxxy,$$

$$\sqrt{(1+myy+ny^4)} = \frac{x}{a} + \frac{y}{a}\sqrt{(1+maa+na^4)} - naxyy.$$

8. Quo has formulas tractatu faciliores reddamus, ponamus

$$\sqrt{(1+maa+na^4)} = \mathfrak{A}$$

similique modo

$$\sqrt{(1+mxx+nx^4)} = \mathfrak{X} \quad \text{et} \quad \sqrt{(1+myy+ny^4)} = \mathfrak{Y}$$

et aequatio nostra erit

$$xx+yy-aa+2\mathfrak{A}xy-naaxxyy=0,$$

unde reperitur

$$y = -\frac{\mathfrak{A}x-a\mathfrak{X}}{1-naaxx} \quad \text{et} \quad x = -\frac{\mathfrak{A}y-a\mathfrak{Y}}{1-naayy};$$

unde patet, si fuerit $y=0$, fore $x=a$; tum vero erunt formulae radicales

$$\sqrt{(1+mxx+nx^4)} = \mathfrak{X} = \frac{y}{a} + \frac{\mathfrak{A}x}{a} - naxxy,$$

$$\sqrt{(1+myy+ny^4)} = \mathfrak{Y} = \frac{x}{a} + \frac{\mathfrak{A}y}{a} - naxyy.$$

9. Quemadmodum autem tam y per x quam x per y exprimere licuit, ita etiam $\mathfrak{Y}$ per solum x et $\mathfrak{X}$ per solum y exprimere licebit. Calculo autem instituto reperietur fore

$$\mathfrak{X} = \frac{(-may+\mathfrak{A}\mathfrak{Y})(1+naayy)-2nay(aa+yy)}{(1-naayy)^2},$$

$$\mathfrak{Y} = \frac{(-max+\mathfrak{A}\mathfrak{X})(1+naaxx)-2nax(aa+xx)}{(1-naaxx)^2}.$$

10. Praecipue autem circa nostram aequationem

$$xx+yy-aa+2\mathfrak{A}xy-naaxxyy=0$$

notari meretur, quod ternae quantitates xx, yy, aa perfecte inter se sint permutabiles. Si enim membrum irrationale ad alteram partem transferatur, ut sit

$$xx+yy-aa-naaxxyy=-2\mathfrak{A}xy,$$

et quadrata sumantur, restituendo pro $\mathfrak{A}^2$ valorem suum $1 + maa + na^4$ prodibit ista aequatio

$$\left.\begin{aligned} &+ x^4 - 2xxyy - 4maaxxyy - 2na^4xxyy + nna^4x^4y^4 \\ &+ y^4 - 2aaxx \qquad\qquad\qquad\quad - 2naax^4yy \\ &+ a^4 - 2aayy \qquad\qquad\qquad\quad - 2naaxxy^4 \end{aligned}\right\} = 0,$$

ubi permutabilitas litterarum a, x, y manifesto in oculos incurrit. In ipsis quidem formulis superioribus, ubi ipsa quantitas a ingreditur, permutabilitas non adeo est manifesta, sed prorsus elucebit, si loco a scribamus $-b$ itemque $\mathfrak{B}$ loco $\mathfrak{A}$; tum enim, quemadmodum erat

$$y = -\frac{x\mathfrak{B} + b\mathfrak{X}}{1 - nbbxx} \quad \text{et} \quad x = -\frac{y\mathfrak{B} + b\mathfrak{Y}}{1 - nbbyy},$$

ita erit

$$b = -\frac{x\mathfrak{Y} + y\mathfrak{X}}{1 - nxxyy}$$

similique modo pro formulis radicalibus seu litteris maiusculis erit

$$\mathfrak{Y} = \frac{(mbx + \mathfrak{B}\mathfrak{X})(1 + nbbxx) + 2nbx(bb + xx)}{(1 - nbbxx)^2},$$

$$\mathfrak{X} = \frac{(mby + \mathfrak{B}\mathfrak{Y})(1 + nbbyy) + 2nby(bb + yy)}{(1 - nbbyy)^2},$$

$$\mathfrak{B} = \frac{(mxy + \mathfrak{X}\mathfrak{Y})(1 + nxxyy) + 2nxy(xx + yy)}{(1 - nxxyy)^2}$$

sicque perfecta permutabilitas perspicitur.

OPERATIO 2

11. Differentiemus nunc nostram aequationem algebraicam assumtam, quae est

$$xx + yy - aa + 2\mathfrak{A}xy - naaxxyy = 0,$$

et aequatio differentialis erit

$$dx(x + \mathfrak{A}y - naaxyy) + dy(y + \mathfrak{A}x - naaxxy) = 0$$

sive

$$\frac{dx}{y + \mathfrak{A}x - naaxxy} + \frac{dy}{x + \mathfrak{A}y - naaxyy} = 0.$$

Ex superioribus autem constat esse

$$y+\mathfrak{A}x-naaxxy=a\mathfrak{X} \quad \text{et} \quad x+\mathfrak{A}y-naaxyy=a\mathfrak{Y},$$

unde aequatio differentialis hanc induet formam

$$\frac{dx}{a\mathfrak{X}}+\frac{dy}{a\mathfrak{Y}}=0$$

sive

$$\frac{dx}{\sqrt{(1+mxx+nx^4)}}+\frac{dy}{\sqrt{(1+myy+ny^4)}}=0.$$

12. Inventa igitur hac aequatione differentiali denotet iste character $\Gamma:x$ integrale $\int\frac{dx}{\mathfrak{X}}$ et character $\Gamma:y$ integrale $\int\frac{dy}{\mathfrak{Y}}$ utroque integrali ita sumto, ut evanescat posito vel $x=0$ vel $y=0$, atque aequationem illam differentialem integrando fiet $\Gamma:x+\Gamma:y=C$. Cum autem sumto $x=0$ fiat etiam $\Gamma:x=0$ et $y=a$, erit constans illa $C=\Gamma:a$, ita ut habeamus hanc aequationem $\Gamma:x+\Gamma:y=\Gamma:a$.

13. Quoniam hic nulla amplius variabilitatis ratio tenetur, patet sumtis binis litteris x et y pro lubitu litteram a ita semper definiri posse, ut fiat

$$\Gamma:a=\Gamma:x+\Gamma:y.$$

Si enim in § 10 loco b scribatur $-a$, sumi debet

$$a=\frac{x\mathfrak{Y}+y\mathfrak{X}}{1-nxxyy},$$

quae comparatio iam casum constituit specialem investigationis generalis, quam suscepimus. Si enim loco x et y scribamus p et q, at r loco a, tum vero $\mathfrak{P}$, $\mathfrak{Q}$ et $\mathfrak{R}$ loco $\mathfrak{X}$, $\mathfrak{Y}$ et $\mathfrak{A}$ atque si sumtis pro lubitu quantitatibus p, q capiatur $r=\frac{p\mathfrak{Q}+q\mathfrak{P}}{1-nppqq}$, tum utique erit $\Gamma:r=\Gamma:p+\Gamma:q$, ita ut hoc casu discrimen illud inter $\Gamma:r$ et summam $\Gamma:p+\Gamma:q$ plane evanescat. Sicque iam evolvimus casum, quo in nostra forma generali

$$\int\frac{Zdz}{\sqrt{(1+mzz+nz^4)}}$$

pro Z sumitur quantitas constans.

OPERATIO 3

14. Quo nunc propius ad nostrum institutum accedamus, sint X et Y tales functiones ipsarum x et y, qualem volumus esse Z ipsius z, et quoniam modo invenimus

$$\frac{dx}{\sqrt{(1+mxx+nx^4)}}+\frac{dy}{\sqrt{(1+myy+ny^4)}}=0,$$

ponamus esse

$$\frac{Xdx}{\sqrt{(1+mxx+nx^4)}}+\frac{Ydy}{\sqrt{(1+myy+ny^4)}}=dV,$$

ita ut, si X et Y essent quantitates constantes, foret $dV=0$. Hinc ergo, si loco $\frac{dy}{\sqrt{(1+myy+ny^4)}}$ scribamus $\frac{-dx}{\sqrt{(1+mxx+nx^4)}}$, fiet

$$dV=\frac{(X-Y)dx}{\sqrt{(1+mxx+nx^4)}} \quad \text{vel etiam} \quad dV=\frac{(Y-X)dy}{\sqrt{(1+myy+ny^4)}}.$$

At si loco radicalium suos valores rationales scribamus, erit

$$dV=\frac{a(X-Y)dx}{y+\mathfrak{A}x-naaxxy} \quad \text{vel} \quad dV=\frac{a(Y-X)dy}{x+\mathfrak{A}y-naaxyy}.$$

15. Cum autem nulla sit ratio, cur istud differentiale dV potius per dx quam per dy exprimamus, consultum erit novam quantitatem in calculum introducere, quae aeque referatur ad x et ad y. Hunc in finem faciamus productum $xy=u$ ac statuamus

$$\frac{dx}{y+\mathfrak{A}x-naaxxy}=-\frac{dy}{x+\mathfrak{A}y-naaxyy}=sdu.$$

Hinc igitur erit

$$dx=sdu(y+\mathfrak{A}x-naaxxy) \quad \text{et} \quad dy=-sdu(x+\mathfrak{A}y-naaxyy),$$

ex quibus colligimus

$$ydx+xdy=sdu(yy-xx)=du,$$

sicque habebimus $s=\frac{1}{yy-xx}$, ita ut habeamus

$$\frac{dx}{y+\mathfrak{A}x-naaxxy}=-\frac{dy}{x+\mathfrak{A}y-naaxyy}=\frac{du}{yy-xx}.$$

Hoc igitur valore substituto nanciscimur

$$dV = \frac{a(X-Y)du}{yy-xx} = \frac{-adu(X-Y)}{xx-yy}.$$

16. Cum autem X et Y sint functiones rationales pares ipsarum x et y, in quibus tantum insunt potestates pares harum litterarum, facile intelligitur formulam $X-Y$ semper esse divisibilem per $xx-yy$ et quotum praeter productum $xy=u$ insuper involvere summam quadratorum $xx+yy$; quamobrem statuamus $xx+yy=t$, et cum aequatio nostra fundamentalis fiat

$$t-aa+2\mathfrak{A}u-naauu=0,$$

ex ea fit

$$t=aa-2\mathfrak{A}u+naauu,$$

ita ut t aequetur functioni rationali ipsius u. Quod si ergo hic valor ubique loco t scribatur, differentiale nostrum quaesitum dV per solam variabilem u exprimetur, ita ut posito $dV=Udu$ semper sit U functio rationalis ipsius u; quae ergo si fuerit integra, tum V aequabitur functioni algebraicae ipsius u, sin autem sit functio fracta, tum integrale $V=\int Udu$ semper per logarithmos et arcus circulares exhiberi poterit. Hoc ergo integrale si ita capiatur, ut evanescat posito $u=xy=0$, id etiam evanescet posito $x=0$ vel $y=0$. Atque hinc integrando impetrabimus

$$\int\frac{Xdx}{\sqrt{(1+mxx+nx^4)}}+\int\frac{Ydy}{\sqrt{(1+myy+ny^4)}}=C+V=C+\int Udu.$$

17. Quod si igitur characteres $\Pi:x$ et $\Pi:y$ denotent valores horum integralium, ita ut utrumque evanescat sumto vel $x=0$ vel $y=0$, quoniam facto $x=0$ per hypothesin fit $y=a$, manifestum est constantem hanc fore $\Pi:a$ sicque aequatio finita resultabit ista

$$\Pi:x+\Pi:y=\Pi:a+\int Udu.$$

18. Accuratius autem in valores huius fractionis U pro quovis casu inquiramus. Ac primo quidem, si sumatur

$$Z=\alpha+\beta zz+\gamma z^4+\delta z^6+\text{etc.},$$

erit simili modo

$$X = \alpha + \beta xx + \gamma x^4 + \delta x^6 + \text{etc.} \quad \text{et} \quad Y = \alpha + \beta yy + \gamma y^4 + \delta y^6 + \text{etc.};$$

quare cum invenerimus

$$dV = Udu = -\frac{adu(X-Y)}{xx-yy},$$

erit

$$U = -\frac{a(X-Y)}{xx-yy} \quad \text{ideoque} \quad U = -\frac{a(\beta(xx-yy) + \gamma(x^4-y^4) + \delta(x^6-y^6))}{xx-yy},$$

unde fit

$$U = -a\beta - a\gamma(xx+yy) - a\delta(x^4 + xxyy + y^4).$$

Cum igitur sit $xx + yy = t$ et $xy = u$, erit

$$U = -a\beta - a\gamma t - a\delta(tt - uu);$$

unde, cum sit $t = aa - 2\mathfrak{A}u + naauu$, calculo subducto fiet

$$\int Udu = -a\beta u - a\gamma\left(aau - \mathfrak{A}uu + \tfrac{1}{3}naau^3\right)$$
$$-a\delta\left(a^4u - 2aa\mathfrak{A}uu + \tfrac{2}{3}na^4u^3 + \tfrac{4}{3}\mathfrak{A}^2u^3 - \tfrac{1}{3}u^3 - n\mathfrak{A}a^2u^4 + \tfrac{1}{5}n^2a^4u^5\right).$$

Atque hinc intelligitur, si functio Z ad altiores potestates exsurgat, quomodo valor integralis ipsius $\int Udu$ inde inveniri queat.

19. Sin autem Z fuerit functio fracta, scilicet

$$Z = \frac{\alpha + \beta zz + \gamma z^4}{\zeta + \eta zz + \theta z^4}$$

hincque

$$X = \frac{\alpha + \beta xx + \gamma x^4}{\zeta + \eta xx + \theta x^4} \quad \text{et} \quad Y = \frac{\alpha + \beta yy + \gamma y^4}{\zeta + \eta yy + \theta y^4},$$

erit

$$X - Y = \frac{(\beta\zeta - \alpha\eta)(xx-yy) + (\gamma\zeta - \alpha\theta)(x^4-y^4) + (\gamma\eta - \beta\theta)x^2y^2(x^2-y^2)}{\zeta\zeta + \zeta\eta(xx+yy) + \zeta\theta(x^4+y^4) + \eta^2x^2y^2 + \eta\theta x^2y^2(xx+yy) + \theta\theta x^4y^4}.$$

Hinc igitur introductis litteris t et u erit

$$\frac{X-Y}{xx-yy} = \frac{\beta\zeta - \alpha\eta + (\gamma\zeta - \alpha\theta)t + (\gamma\eta - \beta\theta)uu}{\zeta\zeta + \zeta\eta t + \zeta\theta(tt - 2uu) + \eta\eta uu + \eta\theta tuu + \theta\theta u^4};$$

quamobrem, cum sit

$$U = -\frac{a(X-Y)}{xx-yy},$$

ob $t = aa - 2\mathfrak{A}u + naauu$ manifestum est integrale formulae $\int Udu$, nisi fuerit algebraicum, semper concessis logarithmis et arcubus circularibus exhiberi posse. Sicque per has tres operationes omnia praestitimus, quibus opus est ad omnia problemata huc spectantia solvenda.

PROBLEMA 1

20. *Si $\Pi : x$ et $\Pi : y$ denotent valores formularum integralium*

$$\int \frac{Xdx}{\sqrt{(1+mxx+nx^4)}} \quad et \quad \int \frac{Ydy}{\sqrt{(1+myy+ny^4)}},$$

ubi X et Y sint functiones pares similes ipsarum x et y, atque dentur binae huiusmodi formulae $\Pi : x$ et $\Pi : y$, invenire tertiam formulam eiusdem generis $\Pi : z$, ut sit $\Pi : z = \Pi : x + \Pi : y + W$, ita ut W sit functio vel algebraica vel per logarithmos et arcus circulares assignabilis.

SOLUTIO

Cum dentur binae quantitates x et y, ex iis formentur radicales

$$\mathfrak{X} = \sqrt{(1+mxx+nx^4)} \quad \text{et} \quad \mathfrak{Y} = \sqrt{(1+myy+ny^4)},$$

ex quibus definiatur quantitas z, eodem modo quo supra litteram a per x et y definire docuimus, ita ut sit

$$z = \frac{x\mathfrak{Y} + y\mathfrak{X}}{1-nxxyy}$$

eiusque valor irrationalis

$$\mathfrak{Z} = \sqrt{(1+mzz+nz^4)} = \frac{(mxy+\mathfrak{X}\mathfrak{Y})(1+nxxyy)+2nxy(xx+yy)}{(1-nxxyy)^2};$$

tum in superioribus formulis ubique loco a et $\mathfrak{A}$ scribamus z et Z et capiatur $U = -\frac{z(X-Y)}{xx-yy}$, quam quantitatem vidimus semper reduci posse ad

functionem ipsius u existente $u=xy$, ac ponatur $V=\int Udu$, in qua integratione quantitates z et $\mathfrak{Z}$ pro constantibus sunt habendae, ita ut littera V spectari possit tanquam functio ipsius $u=xy$, quandoquidem etiam z et $\mathfrak{Z}$ per x et y determinantur. Probe autem teneatur in ista formula integrali solam quantitatem u ut variabilem esse tractandam. Hac igitur quantitate V inventa erit

$$\Pi:x+\Pi:y=\Pi:z+V;$$

unde, cum debeat esse

$$\Pi:z=\Pi:x+\Pi:y+W,$$

patet esse $W=-V$ ideoque quantitatem vel algebraicam vel per logarithmos et arcus circulares assignabilem.

COROLLARIUM 1

21. Totum ergo negotium hic redit ad integrationem formulae Udu existente $u=xy$ et $U=-\frac{z(X-Y)}{xx-yy}$, quam supra vidimus semper per u exprimi posse, siquidem in hac integratione litterae z et $\mathfrak{Z}$ ut quantitates constantes tractentur.

COROLLARIUM 2

22. Cum igitur pro data indole binarum functionum X et Y haec integratio nulla laboret difficultate ipsumque integrale per u, hoc est per xy exprimatur, cuius valorem ex datis quantitatibus x et y semper exhibere liceat, loco quantitatis V scribemus in posterum characterem $\Phi:xy$, unde pro quibusque aliis litteris loco x et y assumtis intelligitur significatus characterum $\Phi:pq$, $\Phi:ab$ etc.

COROLLARIUM 3

23. Hoc igitur charactere recepto si pro datis quantitatibus x et y capiamus $z=\frac{x\mathfrak{Y}+y\mathfrak{X}}{1-nxxyy}$, unde fit

$$\mathfrak{Z}=\frac{(mxy+\mathfrak{X}\mathfrak{Y})(1+nxxyy)+2nxy(xx+yy)}{(1-nxxyy)^2},$$

erit

$$\Pi:z=\Pi:x+\Pi:y-\Phi:xy.$$

PROBLEMA 2

24. *Servatis omnibus characteribus, quos hactenus explicavimus, si dentur ternae formulae $\Pi:p$, $\Pi:q$, $\Pi:r$, invenire quartam eiusdem generis $\Pi:z$, ut fiat*

$$\Pi:z = \Pi:p + \Pi:q + \Pi:r + W,$$

ita ut W sit quantitas algebraica vel per logarithmos arcusve circulares assignabilis.

SOLUTIO

Ex datis binis quantitatibus p et q ideoque etiam $\mathfrak{P}$ et $\mathfrak{Q}$ inde oriundis capiatur

$$x = \frac{p\mathfrak{Q} + q\mathfrak{P}}{1 - nppqq}$$

simulque

$$\mathfrak{X} = \frac{(mpq + \mathfrak{P}\mathfrak{Q})(1 + nppqq) + 2npq(pp + qq)}{(1 - nppqq)^2}.$$

Tum vero etiam colligatur valor characteris $\Phi:pq$ eritque per praecedentia

$$\Pi:x = \Pi:p + \Pi:q - \Phi:pq$$

sive

$$\Pi:p + \Pi:q = \Pi:x + \Phi:pq,$$

quo valore substituto erit

$$\Pi:z = \Pi:x + \Pi:r + \Phi:pq + W.$$

Ex praecedente autem problemate, si loco y hic scribamus r et capiamus

$$z = \frac{x\mathfrak{R} + r\mathfrak{X}}{1 - nrrxx},$$

unde fit

$$\mathfrak{Z} = \frac{(mrx + \mathfrak{R}\mathfrak{X})(1 + nrrxx) + 2nrx(rr + xx)}{(1 - nrrxx)^2},$$

erit

$$\Pi:z = \Pi:x + \Pi:r - \Phi:rx,$$

qua forma cum praecedente collata colligitur

$$W = -\Phi:pq - \Phi:rx,$$

ita ut sit

$$\Pi:z = \Pi:p + \Pi:q + \Pi:r - \Phi:pq - \Phi:rx.$$

PROBLEMA 3

25. *Propositis huiusmodi formulis $\Pi:p$, $\Pi:q$, $\Pi:r$, $\Pi:s$ invenire quintam eiusdem generis $\Pi:z$, ut fiat*

$$\Pi:z = \Pi:p + \Pi:q + \Pi:r + \Pi:s + W,$$

ita ut W sit quantitas algebraica vel per logarithmos arcusve circulares assignabilis.

SOLUTIO

Ex datis binis p et q quaeratur x, ut sit

$$x = \frac{p\mathfrak{Q} + q\mathfrak{P}}{1 - nppqq},$$

item

$$\mathfrak{X} = \frac{(mpq + \mathfrak{P}\mathfrak{Q})(1 + nppqq) + 2npq(pp + qq)}{(1 - nppqq)^2},$$

eritque

$$\Pi:x = \Pi:p + \Pi:q - \Phi:pq.$$

Simili modo ex binis datis r et s quaeratur y, ut sit

$$y = \frac{r\mathfrak{S} + s\mathfrak{R}}{1 - nrrss},$$

eritque

$$\mathfrak{Y} = \frac{(mrs + \mathfrak{R}\mathfrak{S})(1 + nrrss) + 2nrs(rr + ss)}{(1 - nrrss)^2},$$

tum vero

$$\Pi:y = \Pi:r + \Pi:s - \Phi:rs.$$

Nunc denique ex inventis x et y sumatur

$$z = \frac{x\mathfrak{Y} + y\mathfrak{X}}{1 - nxxyy} \quad \text{et} \quad \mathfrak{Z} = \frac{(mxy + \mathfrak{X}\mathfrak{Y})(1 + nxxyy) + 2nxy(xx + yy)}{(1 - nxxyy)^2}$$

eritque

$$\Pi:z = \Pi:x + \Pi:y - \Phi:xy.$$

Quodsi ergo loco $\Pi:x$ et $\Pi:y$ valores modo inventi substituantur, fiet

$$\Pi:z = \Pi:p + \Pi:q + \Pi:r + \Pi:s - \Phi:pq - \Phi:rs - \Phi:xy.$$

COROLLARIUM 1

26. Hinc iam abunde intelligitur, si proponantur quotcunque huiusmodi formulae, quemadmodum novam eiusdem generis $\Pi : z$ investigari oporteat, quae ab illis iunctim sumtis discrepet quantitate algebraica vel per logarithmos arcusve circulares assignabili.

COROLLARIUM 2

27. Quod si omnes illae formulae fuerint inter se aequales earumque numerus $= \lambda$, semper nova formula $\Pi : z$ inveniri poterit, ut sit

$$\Pi : z = \lambda \Pi : p + W$$

existente W quantitate vel algebraica vel per logarithmos arcusve circulares assignabili. Quin etiam duabus huiusmodi formulis $\Pi : p$ et $\Pi : q$ propositis inveniri poterit $\Pi : z$, ut sit

$$\Pi : z = \lambda \Pi : p + \mu \Pi : q + W.$$

SCHOLION

28. Hoc igitur modo non solum principia et fundamenta, quibus hoc argumentum innititur, succincte ac dilucide mihi quidem exposuisse videor, sed hoc argumentum etiam multo latius amplificavi, quam adhuc est factum. Semper autem maxime est optandum, ut via magis directa detegatur, quae ad easdem investigationes perducat. Nemo enim certe dubitabit, quin hinc maxima in universam Analysin incrementa essent redundatura.

APPLICATIO AD QUANTITATES TRANSCENDENTES IN FORMA $\int \frac{dz(\alpha + \beta zz)}{\sqrt{(1 + mzz + nz^4)}} = \Pi : z$ CONTENTAS

29. Cum igitur hic sit $Z = \alpha + \beta zz$, propositis duabus formulis huius generis $\Pi : x$ et $\Pi : y$ sumtoque

$$z = \frac{x\mathfrak{Y} + y\mathfrak{X}}{1 - nxxyy} \quad \text{hincque} \quad \mathfrak{Z} = \frac{(mxy + \mathfrak{X}\mathfrak{Y})(1 + nxxyy) + 2nxy(x^2 + y^2)}{(1 - nxxyy)^2}$$

ex § 18, ubi $u = xy$ et $a = z$, erit

$$\Pi : z = \Pi : x + \Pi : y + \beta xyz,$$

ita ut character ante adhibitus $\Phi : xy$ hoc casu accipiat valorem βxyz. Huius igitur regulae ope propositis duabus huiusmodi formulis $\Pi : x$ et $\Pi : y$ tertia $\Pi : z$ semper reperiri potest, quae a summa illarum differat quantitate algebraica βxyz.

30. Ponamus igitur quotcunque huiusmodi formulas transcendentes proponi

$$\Pi : a, \quad \Pi : b, \quad \Pi : c, \quad \Pi : d, \quad \Pi : e, \quad \Pi : f, \quad \Pi : g \quad \text{etc.}$$

et ex singulis quantitatibus a, b, c, d etc. colligi valores irrationales litteris germanicis insignitas

$$\begin{aligned} \mathfrak{A} &= \sqrt{(1 + maa + na^4)}, & \mathfrak{B} &= \sqrt{(1 + mbb + nb^4)}, \\ \mathfrak{C} &= \sqrt{(1 + mcc + nc^4)}, & \mathfrak{D} &= \sqrt{(1 + mdd + nd^4)}, \\ &\text{etc.} & &\text{etc.} \end{aligned}$$

semper nova formula eiusdem generis exhiberi poterit, quae a summa earum discrepet quantitate algebraica, quantuscunque etiam fuerit earum formularum datarum numerus. Operationes autem ad hunc finem perducentes commodissime sequenti modo instituentur.

31. Primo scilicet ex binis datarum a et b quaeratur p, ut sit

$$p = \frac{a\mathfrak{B} + b\mathfrak{A}}{1 - naabb} \quad \text{et} \quad \mathfrak{P} = \frac{(mab + \mathfrak{A}\mathfrak{B})(1 + naabb) + 2nab(aa + bb)}{(1 - naabb)^2}.$$

Deinde ex hac quantitate p cum datarum tertia c iuncta definiatur q, ut sit

$$q = \frac{p\mathfrak{C} + c\mathfrak{P}}{1 - nccpp} \quad \text{et} \quad \mathfrak{Q} = \frac{(mcp + \mathfrak{C}\mathfrak{P})(1 + nccpp) + 2ncp(cc + pp)}{(1 - nccpp)^2}.$$

Tertio ex hac quantitate q cum quarta datarum d iuncta quaeratur r, ut sit

$$r = \frac{q\mathfrak{D} + d\mathfrak{Q}}{1 - nddqq} \quad \text{et} \quad \mathfrak{R} = \frac{(mdq + \mathfrak{D}\mathfrak{Q})(1 + nddqq) + 2ndq(dd + qq)}{(1 - nddqq)^2}.$$

Quarto ex ista quantitate r cum quinta datarum e iuncta definiatur s, ut sit

$$s = \frac{r\mathfrak{E} + e\mathfrak{R}}{1 - neerr} \quad \text{et} \quad \mathfrak{S} = \frac{(mer + \mathfrak{E}\mathfrak{R})(1 + neerr) + 2ner(ee + rr)}{(1 - neerr)^2}.$$

Haeque operationes continuentur, donec omnes quantitates datae in computum fuerint ductae.

32. His autem omnibus valoribus inventis sequentes comparationes desideratae ordine ita se habebunt

$$\text{I.} \quad \Pi : p = \Pi : a + \Pi : b + \beta abp,$$

$$\text{II.} \quad \Pi : q = \Pi : a + \Pi : b + \Pi : c + \beta abp + \beta cpq,$$

$$\text{III.} \quad \Pi : r = \Pi : a + \Pi : b + \Pi : c + \Pi : d + \beta abp + \beta cpq + \beta dqr,$$

$$\text{IV.} \quad \Pi : s = \Pi : a + \Pi : b + \Pi : c + \Pi : d + \Pi : e \\ + \beta abp + \beta cpq + \beta dqr + \beta ers,$$

$$\text{V.} \quad \Pi : t = \Pi : a + \Pi : b + \Pi : c + \Pi : d + \Pi : e + \Pi : f \\ + \beta abp + \beta cpq + \beta dqr + \beta ers + \beta fst$$

etc.

33. Cum igitur ista formula transcendens

$$\Pi : z = \int \frac{dz(\alpha + \beta zz)}{\sqrt{(1 + mzz + nz^4)}}$$

in se contineat arcus omnium sectionum conicarum a vertice sumtos, harum formularum ope, quotcunque proponantur arcus in quavis sectione conica, qui omnes a vertice sint sumti, semper novus in eadem sectione conica arcus pariter a vertice abscindi poterit, qui a summa illorum arcuum datorum discrepet quantitate algebraice assignabili. Quin etiam nihil impedit, quo minus aliqui inter arcus datos capiantur negativi, quandoquidem iam annotavimus esse $\Pi : (-z) = -\Pi : z$, ita ut nostra determinatio etiam accommodari possit ad arcus inter terminos quoscunque interceptos. Hocque modo tractatio, quam nuper circa comparationem talium arcuum dedi, multo generalior reddi poterit.

34. Ceterum, quemadmodum hoc casu, quo sumsimus $Z = \alpha + \beta zz$, character supra usurpatus $\Phi : xy$ abiit in βxyz, dum scilicet ex binis quantitatibus x et y secundum praecepta data tertia z determinatur, ita etiam, quaecunque alia functio loco Z adhibeatur, quoniam posuimus

$$\Phi : xy = -a\int \frac{(X - Y)\,du}{xx - yy}$$

existente $u = xy$, integratione absoluta functio inde resultans tantum quantitatem u cum litteris a et $\mathfrak{A}$ continebit, quandoquidem littera t ita exprimebatur

$$t = aa - 2\mathfrak{A}u + naauu,$$

cum invento integrali ubique loco u scribatur xy, at vero loco a et $\mathfrak{A}$ litterae z et $\mathfrak{Z}$; atque hoc modo obtinebitur valor characteris $\Phi : xy$ pro quovis casu proposito, quae functio, nisi fuerit algebraica, semper per logarithmos et arcus circulares exhiberi poterit, siquidem, uti assumsimus, littera Z fuerit functio rationalis par ipsius z.

UBERIOR EVOLUTIO COMPARATIONIS QUAM INTER ARCUS SECTIONUM CONICARUM INSTITUERE LICET

Commentatio 582 indicis ENESTROEMIANI
Acta academiae scientiarum Petropolitanae 1781: II (1785), p. 23—44

1. Novum fere etiamnunc est argumentum et minime adhuc satis exploratum, quod in omni sectione conica sumto pro lubitu arcu quocunque ab alio quovis puncto eiusdem curvae semper arcum rescindere liceat, qui ab illo arcu differat quantitate geometrice assignabili. Ita si in sectione conica AB (Fig. 1) pro lubitu accipiatur arcus EF, tum ab alio quocunque puncto M semper rescindi potest arcus MN, ita ut differentia inter arcus EF et MN algebraice assignari queat; hocque adeo duplici modo praestare licet, prouti a puncto M arcum desideratum vel antrorsum, uti MN, vel retrorsum, uti Mn, abscindere velimus.

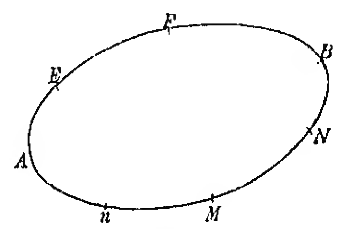

Fig. 1.

Quod si sectio conica fuerit circulus, res ex primis elementis adeo est manifesta, ubi quidem differentia inter binos illos arcus necessario est nulla. Pro parabola autem idem iam dudum a BERNOULLIIS est ostensum, quandoquidem quilibet arcus parabolicus per aggregatum ex quantitate algebraica et logarithmica exprimitur. Quod vero ad ellipsin et hyperbolam attinet, quarum rectificationem neque per arcus circulares neque per logarithmos expedire licet, talis comparatio vires Analyseos penitus superare videbatur, donec ab Illustrissimo Comite FAGNANO prima principia fuere patefacta, quae ad hunc scopum deducerent et quae deinceps accuratius sum prosecutus, ita ut ista investigatio multo latius sit extensa multoque facilius ad innumeras alias speculationes accommodari queat. Interim tamen operationes, quibus

hoc negotium absolvitur, tantopere ab operationibus analyticis solitis recedunt, ut ad singulare calculi genus referendae videantur, cum nequidem veritas istiusmodi comparationum more solito per calculum ostendi possit.

2. Foecundissimum autem hoc argumentum in pluribus dissertationibus Commentariis Academiae Petropolitanae fusius sum persecutus atque adeo plures methodos detexi, quae ad eundem finem perducere valeant, quae autem nihilominus ita sunt comparatae, ut tota ista investigatio adhuc penitus nova et a vulgari calculo analytico plurimum recedens habenda videatur. Huic eidem argumento etiam sectionem peculiarem in *Institutionibus* meis *Calculi integralis* tribuendam censui, ubi duobus capitibus[1]) hoc argumentum prorsus novum a pag. 421 usque ad pag. 493 sum complexus, unde praecipua momenta ad rectificationem sectionum conicarum spectantia depromam, quae in sequente theoremate generali sum comprehensurus.

THEOREMA GENERALE

3. *Si character $\Pi : z$ denotet valorem formulae integralis*

$$\int \frac{dz(L+Mzz+Nz^4)}{\sqrt{(A+Czz+Ez^4)}}$$

ita sumtum, ut evanescat posito $z=0$, semper ternas huiusmodi formulas $\Pi : p$, $\Pi : q$, $\Pi : r$ ita inter se comparare licet, ut sit

$$\Pi : p + \Pi : q + \Pi : r = \frac{Mpqr}{\sqrt{A}} + \frac{Npqr}{2A\sqrt{A}}\Big(A(pp+qq+rr) - \frac{1}{3}Eppqqrr\Big),$$

si modo inter quantitates p, q et r ista relatio stabiliatur, ut sit

$$r = \frac{-p\sqrt{A(A+Cqq+Eq^4)} - q\sqrt{A(A+Cpp+Ep^4)}}{A-Eppqq},$$

unde simili modo patet fore

$$p = \frac{-q\sqrt{A(A+Crr+Er^4)} - r\sqrt{A(A+Cqq+Eq^4)}}{A-Eqqrr},$$

$$q = \frac{-p\sqrt{A(A+Crr+Er^4)} - r\sqrt{A(A+Cpp+Ep^4)}}{A-Epprr}.$$

1) *Institutionum Calculi integralis* vol. 1 sect. 2 cap. V et VI; LEONHARDI EULERI *Opera omnia*, series I, vol. 11. A. K.

DILUCIDATIONES

4. Cum sit

$$\Pi : z = \int \frac{dz(L + Mzz + Nz^4)}{\sqrt{(A + Czz + Ez^4)}}$$

integrali ita sumto, ut evanescat posito $z = 0$, patet fore $\Pi : 0 = 0$; tum vero, quoniam sumto z negativo valor formulae integralis etiam fit negativus, patet fore $\Pi : (-z) = -\Pi : z$, unde, si quantitatum p, q, r una, veluti p, fuerit negativa, tum in relatione assignata loco $\Pi : p$ scribi debet $-\Pi : p$. Ceterum manifestum est hanc formulam integralem maxime fore transcendentem, cum neque per logarithmos neque per quadraturam circuli expediri possit, ita ut ista quantitas $\Pi : z$ per nullas formulas in Analysi receptas exhiberi queat. Paucissimi quidem casus hinc sunt excipiendi, quibus est vel $E = 0$ (hoc enim casu formula per logarithmos vel arcus circulares assignari posset, quod idem eveniret, si esset $A = 0$), vel quando quantitas $A + Czz + Ez^4$ fuerit quadratum, quo casu iterum integratio ut ante succederet, vel denique, si litterae L, M et N ita fuerint comparatae, ut formula proposita algebraicum accipiat integrale, cuius forma erit $\alpha z\sqrt{(A + Czz + Ez^4)}$. Quia enim eius differentiale est

$$\frac{\alpha dz(A + 2Czz + 3Ez^4)}{\sqrt{(A + Czz + Ez^4)}},$$

si fuerit $L = \alpha A$, $M = 2\alpha C$ et $N = 3\alpha E$, formula $\Pi : z$ utique huic quantitati algebraicae $\alpha z\sqrt{(A + Czz + Ez^4)}$ aequabitur.

5. Quemadmodum hoc argumentum in variis dissertationibus tractavi, in formula integrali numeratorem $L + Mzz + Nz^4$ ulterius per potestates pares, quousque libuerit, continuare licuisset eius loco ponendo

$$L + Mzz + Nz^4 + Oz^6 + Pz^8 + Qz^{10} + \text{etc.};$$

verum quia quaelibet potestas ad binas praecedentes facile reduci potest, tali extensione carere poterimus; semper enim statui potest

$$\int \frac{z^{n+4}dz}{\sqrt{(A + Czz + Ez^4)}} = \alpha z^{n+1}\sqrt{(A + Czz + Ez^4)} + \int \frac{dz(\mathfrak{A}z^n + \mathfrak{B}z^{n+2})}{\sqrt{(A + Czz + Ez^4)}}.$$

Erit enim

$$\alpha = \frac{1}{(n+3)E}, \quad \mathfrak{A} = \frac{-(n+1)A}{(n+3)E} \quad \text{et} \quad \mathfrak{B} = \frac{-(n+2)C}{(n+3)E}.$$

Hinc igitur sumto $n=0$ fiet

$$\int\frac{z^4dz}{\sqrt{(A+Czz+Ez^4)}}=\frac{1}{3E}z\sqrt{(A+Czz+Ez^4)}-\frac{1}{3E}\int\frac{(A+2Czz)dz}{\sqrt{(A+Czz+Ez^4)}},$$

quamobrem hic etiam in nostra formula integrali terminum Nz^4 omittere potuissemus.

6. Cum igitur non obstante transcendentia formulae $\Pi:z$ ternas huiusmodi formulas $\Pi:p$, $\Pi:q$ et $\Pi:r$ semper ita inter se comparare liceat, ut earum summa $\Pi:p+\Pi:q+\Pi:r$ aequetur quantitati algebraicae

$$\frac{Mpqr}{\sqrt{A}}+\frac{Npqr}{2A\sqrt{A}}\left(A(p^2+q^2+r^2)-\frac{1}{3}Ep^2q^2r^2\right),$$

si modo inter tres quantitates p, q, r ea relatio accipiatur, quae in theoremato est praescripta, haec relatio eo magis est notatu digna, quod ternae litterae p, q, r in illam formam aequaliter ingrediantur, ita ut prorsus inter se pro lubitu permutari queant. Cum igitur nullae adhuc huiusmodi relationes in Analysi sint consideratae, haec investigatio utique maxime ardua est censenda ac nullum est dubium, quin plurima insuper mysteria analytica altioris indaginis in se involvat, quae eo magis abscondita videntur, quod a consuetis Analyseos operationibus maxime recedunt.

7. Ternarum autem quantitatum illarum p, q, r binas pro lubitu assumere licet, dummodo tertiae is valor tribuatur, qui in theoremate assignatus est; quae relatio quo concinnius exprimi queat, statuamus brevitatis gratia

$$\sqrt{A(A+Cpp+Ep^4)}=P,$$

$$\sqrt{A(A+Cqq+Eq^4)}=Q$$

et

$$\sqrt{A(A+Crr+Er^4)}=R;$$

tum enim, si binae p et q fuerint datae, erit $r=\frac{-pQ-qP}{A-Eppqq}$; sin autem litterae p et r fuerint datae, erit $q=\frac{-pR-rP}{A-Epprr}$; sin autem binae q et r fuerint datae, erit $p=\frac{-qR-rQ}{A-Eqqrr}$.

8. Pro quovis autem horum casuum etiam plurimum intererit valores litterarum maiuscularum P, Q et R per binas reliquas expressisse. Ponamus igitur binas litteras p et q ideoque etiam P et Q esse datas, ita ut sit $r = \frac{-pQ - qP}{A - Eppqq}$; unde, si immediate valorem ipsius R quaerere vellemus, in maximas tricas calculi illaberemur, ad quas evitandas ex tertia relatione quaeramus valorem ipsius R, qui erit

$$R = \frac{-rQ - p(A - Eqqrr)}{q};$$

ubi si loco r et rr valores substituantur et loco quadratorum P^2 et Q^2 sui valores scribantur, tandem reperietur

$$R = \frac{(ACpq + PQ)(A + Eppqq) + 2AAEpq(pp + qq)}{(A - Eppqq)^2}.$$

Simili modo ex datis p et r cum P et R erit

$$Q = \frac{(ACpr + PR)(A + Epprr) + 2AAEpr(pp + rr)}{(A - Epprr)^2}$$

ac denique ex datis q et r cum Q et R fiet

$$P = \frac{(ACqr + QR)(A + Eqqrr) + 2AAEqr(qq + rr)}{(A - Eqqrr)^2}.$$

9. Cum igitur isti valores tantopere sint complicati atque adeo duplicem irrationalitatem involvant, maxime mirum videbitur, quomodo eos in formulis differentialibus substituere, multo magis autem, quomodo inde ad formulas tractabiles atque adeo integrabiles perveniri queat. Interim tamen hae tantae difficultates haud mediocriter sublevabuntur, si differentiale quantitatis r ex formula $r = \frac{-pQ - qP}{A - Eppqq}$ evolvamus.

10. Qui labor quo facilius succedat, primo tantum quantitatem p pro variabili habeamus, quandoquidem differentiale ex variabilitate ipsius q oriundum sponte definitur. Sint igitur solae litterae p et P variabiles eritque

$$dr = \frac{-Qdp - qdP}{A - Eppqq} - \frac{2Epqqdp(pQ + qP)}{(A - Eppqq)^2};$$

quia igitur

$$dP = \frac{ACpdp + 2AEp^3dp}{\sqrt{A(A + Cpp + Ep^4)}},$$

calculo subducto reperietur tandem

$$dr = \frac{-dp(ACpq + PQ)(A + Eppqq) - 2AAEpqdp(pp + qq)}{(A - Eppqq)^2 P}$$

similique modo ob variabilitatem ipsius q colligetur

$$dr = \frac{-dq(ACpq + PQ)(A + Eppqq) - 2AAEpqdq(pp + qq)}{(A - Eppqq)^2 Q},$$

quae duae expressiones iunctim sumtae differentiale completum quantitatis r praebebunt.

11. Hic autem imprimis notari meretur, quod in utraque formula differentiali pro dp et dq numerator prorsus idem prodierit atque adeo ille penitus cum valore pro R supra invento congruat (vide § 8). Hoc igitur valore substituto completum differentiale quantitatis r erit

$$dr = -\frac{Rdp}{P} - \frac{Rdq}{Q},$$

ita ut sit

$$\frac{dr}{R} = -\frac{dp}{P} - \frac{dq}{Q}.$$

Hinc igitur loco P, Q, R suos valores substituendo et per $\sqrt{A}$ multiplicando erit

$$\frac{dp}{\sqrt{(A + Cpp + Ep^4)}} + \frac{dq}{\sqrt{(A + Cqq + Eq^4)}} + \frac{dr}{\sqrt{(A + Crr + Er^4)}} = 0,$$

unde sequitur fore

$$\int \frac{dp}{\sqrt{(A + Cpp + Ep^4)}} + \int \frac{dq}{\sqrt{(A + Cqq + Eq^4)}} + \int \frac{dr}{\sqrt{(A + Crr + Er^4)}} = 0,$$

siquidem singula integralia ita capiantur, ut evanescant posito $p = 0$, $q = 0$ et $r = 0$.

12. Hac insigni proprietate inventa inquiramus porro, quemadmodum inde principalis relatio inter formulas $\Pi : p$, $\Pi : q$ et $\Pi : r$ ostendi queat; quod quo facilius fieri possit, in numeratoribus formularum nostrarum integralium suma-

mus $N=0$ atque ostendi oportebit istam aequationem integralem semper locum habere

$$\int\frac{dp(L+Mpp)}{P}+\int\frac{dq(L+Mqq)}{Q}+\int\frac{dr(L+Mrr)}{R}=\frac{Mpqr}{A}.$$

Quodsi iam loco $\frac{dr}{R}$ scribamus $-\frac{dp}{P}-\frac{dq}{Q}$, aequatio ista hanc induet formam

$$M\int\frac{dp(pp-rr)}{P}+M\int\frac{dq(qq-rr)}{Q}=\frac{Mpqr}{A},$$

sive ad differentialia descendendo ostendi debet hanc aequationem veritati esse consentaneam

$$\frac{dp(pp-rr)}{P}+\frac{dq(qq-rr)}{Q}=\frac{pqdr}{A}+\frac{prdq}{A}+\frac{qrdp}{A}.$$

Quodsi ergo in dextra parte scribamus loco dr valorem $-\frac{Rdp}{P}-\frac{Rdq}{Q}$, demonstrandum est fore

$$\frac{dp(pp-rr)}{P}+\frac{dq(qq-rr)}{Q}=\frac{qdp(rP-pR)}{AP}+\frac{pdq(rQ-qR)}{AQ}$$

sive

$$\frac{dp(App-Arr-qrP+pqR)}{AP}+\frac{dq(Aqq-Arr-prQ+pqR)}{AQ}=0.$$

13. Cum igitur haec aequalitas subsistere debeat, quicunque valores binis variabilibus p et q tribuantur, necesse est, ut utraque pars seorsim nihilo aequetur; quocirca ostendi debet fore tam

$$App-Arr-qrP+pqR=0$$

quam

$$Aqq-Arr-prQ+pqR=0.$$

Harum aequationum posterior a priore subtracta relinquit

$$A(pp-qq)-r(qP-pQ)=0;$$

ubi si loco r valor substituatur, fit

$$A(pp-qq)+\frac{qqPP-ppQQ}{A-Eppqq}=0.$$

Est vero

$$qqPP - ppQQ = (qq - pp)AA - 4Ippqq,$$

unde aequalitas manifesto patet. Tantum igitur superest, ut veritas alterius doceatur. Supra autem vidimus esse

$$R = \frac{rQ + pA + Ipqr}{q},$$

qui valor in priore aequatione substitutus praebet

$$Arr - r(qP + pQ) + Ippqqr = 0,$$

deinde vero, quia

$$qP + pQ = rA + Ippqq,$$

hoc valore substituto resultat

$$Arr - rrA - Ippqqr + Ippqqr = 0,$$

cuius veritas est manifesta.

14. Hoc igitur modo ex nostris formulis veritatem theorematis generalis pro casu $N = 0$ per multas quidem ambages elicere possumus. Facile autem intelligitur, si etiam litterae N rationem habere vellemus, demonstrationem difficillimis calculis fore involutam, quos vix quisquam esset suscepturus, nisi iam ante de veritate asserti nostri fuisset convictus. Ideo maxime igitur nostrum theorema omni attentione et admiratione dignum est censendum, quod per consueta Analyseos artificia vix ulla via pateat eius demonstrationem in genere concinnandi, multo minus hac solutione veritatem a priori investigandi.

APPLICATIO
AD SECTIONES CONICAS

15. Consideremus igitur semiellipsin ACB (Fig. 2), cuius centrum sit in O, ac ponatur semiaxis transversus $AO = BO = a$ et semiaxis coniugatus $OC = c$.

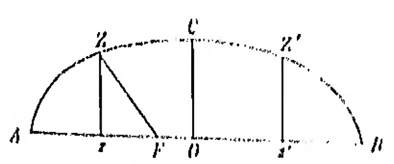

Fig. 2.

Tum vero ducta applicata quacunque $Zz = z$ denotet nostra formula $\Pi : z$ arcum ellipsis AZ illi applicatae respondentem; unde patet, si fuerit $z = 0$, fore etiam $\Pi : z = 0$, at sumta $Zz = OC = c$ erit

$\Pi : c = AC$, scilicet quadranti elliptico aequale. Hinc autem intelligitur eidem applicatae Zz innumerabiles respondere arcus ellipticos; praeter minimum enim AZ ipsi convenient arcus $4\Pi : c + AZ$, item $8\Pi : c + AZ$, $12\Pi : c + AZ$ etc. Praeterea vero, quia ex altera parte etiam datur talis applicata $Z'z'$, ei quoque conveniet arcus

$$AZ' = 2\Pi : c - AZ$$

similique modo etiam $6\Pi : c - AZ$, $10\Pi : c - AZ$ etc. sicque ista formula $\Pi : z$ erit functio infinitiformis ipsius z, scilicet in ellipsi; nam in hyperbola omnes isti valores praeter unum vel duos evadent imaginarii.

16. Pro arcu igitur AZ analytice exprimendo vocetur abscissa $Oz = v$, et cum sit ex natura ellipsis

$$\frac{vv}{aa} + \frac{zz}{cc} = 1,$$

erit

$$v = \frac{a}{c}\sqrt{(cc - zz)}$$

hincque

$$dv = -\frac{azdz}{c\sqrt{(cc - zz)}},$$

unde colligitur elementum arcus AZ

$$\sqrt{(dv^2 + dz^2)} = dz\sqrt{1 + \frac{aazz}{cc(cc - zz)}} = dz\sqrt{\frac{c^4 + (aa - cc)zz}{cc(cc - zz)}},$$

quocirca habebimus

$$\Pi : z = \int \frac{dz\sqrt{(c^4 + (aa - cc)zz)}}{c\sqrt{(cc - zz)}}.$$

17. Cum igitur in genere posuissemus

$$\Pi : z = \int \frac{dz(L + Mzz + Nz^4)}{\sqrt{(A + Czz + Ez^4)}},$$

ante omnia nostram formulam ad eandem formam reducamus, dum scilicet eius numeratorem et denominatorem multiplicamus per $\sqrt{(c^4 + (aa - cc)zz)}$; tum autem prodibit

$$\Pi : z = \int \frac{dz(c^4 + (aa - cc)zz)}{c\sqrt{(cc - zz)(c^4 + (aa - cc)zz)}},$$

unde patet pro hoc casu fore $L = c^4$, $M = aa - cc$ et $N = 0$, deinde vero $A = c^6$, $C = c^4(aa - 2cc)$ et $E = -cc(aa - cc)$, unde, si ut supra brevitatis gratia ponamus

$$Z = \sqrt{A(A + Czz + Ez^4)},$$

erit

$$Z = c^5\sqrt{(cc - zz)(c^4 + aazz - cczz)}.$$

His igitur formulis eodem modo uti conveniet, uti in genere est monstratum.

18. Quo has formulas concinniores reddamus, loco litterae c introducamus semiparametrum ellipsis, qui sit $= b$, et cum sit $cc = ab$, hoc pacto [illegible]

$$Z = a^3bb\sqrt{b(ab - zz)(abb + [illegible])}$$

hincque fiet ipsa formula

$$\Pi : z = \int \frac{dz\sqrt{(abb + (a - b)zz)}}{\sqrt{b(ab - zz)}}.$$

Praeterea vero erit $L = aabb$, $M = a(a - b)$, [illegible] et $E = -aab(a - b)$. Loco semiaxis transversi a nuper introducamus excentricitatem, quae sit $= n$, et quia ex elementis constat [illegible], hoc valore substituto fiet

$$Z = \frac{b^5}{(1 - nn)^4}\sqrt{(b^3 - b(1 - nn)zz)(b^2 + [illegible]nnzz)} [illegible] \frac{[illegible]}{1 - nn}$$

Vel potius hanc totam reductionem a principio repetamus, et cum sit

$$\Pi : z = \int \frac{dz\sqrt{(bb + nnzz)}}{\sqrt{(bb - (1 - nn)zz)}},$$

hac ad formam generalem reducta fit

$$\Pi : z = \int \frac{dz(bb + nnzz)}{\sqrt{(bb + nnzz)(bb - (1 - nn)zz)}}$$

ideoque comparatio praebet $L = bb$, $M = nn$, $N = 0$, $A = b^4$, $C = bb(2nn - 1)$ et $E = -nn(1 - nn)$; tum vero erit

$$Z = bb\sqrt{(bb + nnzz)(bb - (1 - nn)zz)}.$$

Atque nunc haec formula aeque valet pro omnibus sectionibus conicis. Quando enim $n<1$, habebitur ellipsis; casu $n=1$ parabola; at si $n>1$, prodit hyperbola; pro circulo autem erit $n=0$.

19. Statuantur nunc ternae applicatae p, q, r indeque deriventur valores derivati

$$P = bb\sqrt{(bb+nnpp)(bb-(1-nn)pp)},$$
$$Q = bb\sqrt{(bb+nnqq)(bb-(1-nn)qq)},$$
$$R = bb\sqrt{(bb+nnrr)(bb-(1-nn)rr)};$$

tum vero ex binis p et q tertia r ita determinetur, ut sit

$$r = \frac{-pQ-qP}{b^4+nn(1-nn)ppqq},$$

eritque

$$R = \frac{(b^6(2nn-1)pq+PQ)(b^4-nn(1-nn)ppqq)-2b^8nn(1-nn)pq(pp+qq)}{(b^4+nn(1-nn)ppqq)^2},$$

quibus positis habebitur sequens comparatio ternorum arcuum

$$\varPi:p+\varPi:q+\varPi:r = \frac{nnpqr}{bb},$$

ubi binos arcus $\varPi:p$ et $\varPi:q$ pro lubitu assumere licet; hinc enim semper assignari poterit tertius $\varPi:r$, ut omnium summa fiat quantitas algebraica, dummodo notetur horum arcuum semper unum duosve fore negativos, cum sit $\varPi:(-z)=-\varPi:z$.

TRANSLATIO FORMULARUM PRAECEDENTIUM AD ALTERUTRUM FOCUM SECTIONIS CONICAE

20. Sit nunc F alteruter focus nostrae ellipsis seu sectionis conicae in genere, qui quidem vertici A sit propior; atque ex elementis constat posito angulo $AFZ=\varphi$ tum fore distantiam

$$FZ = \frac{b}{1+n\cos.\varphi},$$

unde colligitur applicata

$$Zz = z = \frac{b\sin.\varphi}{1+n\cos.\varphi},$$

ita ut nunc sit arcus

$$AZ = \Pi : z = \Pi : \frac{b \sin. \varphi}{1 + n \cos. \varphi},$$

qui ergo, cum nunc spectetur ut functio anguli φ, designetur hoc charactere $AZ = \Gamma : \varphi$, ita ut sit

$$\Pi : z = \Pi : \frac{b \sin. \varphi}{1 + n \cos. \varphi} = \Gamma : \varphi.$$

Videamus igitur, quomodo iste arcus per angulum φ exprimatur; constat autem posita distantia $FZ = v$ fore arcum $AZ = \int \sqrt{(dv^2 + vvd\varphi^2)}$; quare cum sit

$$v = \frac{b}{1 + n \cos. \varphi},$$

erit

$$dv = \frac{nbd\varphi \sin. \varphi}{(1 + n \cos. \varphi)^2},$$

unde fit

$$dv^2 = \frac{nnbbd\varphi^2 \sin. \varphi^2}{(1 + n \cos. \varphi)^4},$$

cui si addatur

$$vvd\varphi^2 = \frac{bbd\varphi^2}{(1 + n \cos. \varphi)^2},$$

erit summa

$$= \frac{bbd\varphi^2(1 + 2n \cos. \varphi + nn)}{(1 + n \cos. \varphi)^4}$$

sicque erit arcus

$$AZ = \Pi : z = \Gamma : \varphi = \int \frac{bd\varphi}{(1 + n \cos. \varphi)^2} \sqrt{(1 + 2n \cos. \varphi + nn)}$$

$$= b \int \frac{d\varphi \sqrt{(1 + nn + 2n \cos. \varphi)}}{(1 + n \cos. \varphi)^2}.$$

Hinc autem porro colligetur

$$Z = b^4 \sqrt{\left(1 + \frac{nn \sin. \varphi^2}{(1 + n \cos. \varphi)^2}\right)\left(1 + \frac{(nn - 1) \sin. \varphi^2}{(1 + n \cos. \varphi)^2}\right)}$$

sive

$$Z = \frac{b^4}{(1 + n \cos. \varphi)^2} \sqrt{(1 + nn + 2n \cos. \varphi)(nn + 2n \cos. \varphi + \cos. \varphi^2)}$$

sive

$$Z = \frac{b^4(n + \cos. \varphi) \sqrt{(1 + nn + 2n \cos. \varphi)}}{(1 + n \cos. \varphi)^2}.$$

21. Quodsi iam in calculum introducamus ternas applicatas p, q et r, quibus respondeant anguli ad focum ζ, η et θ, ita ut sit

$$p = \frac{b \sin.\zeta}{1+n\cos.\zeta}, \quad q = \frac{b\sin.\eta}{1+n\cos.\eta} \quad \text{et} \quad r = \frac{b\sin.\theta}{1+n\cos.\theta},$$

tum vero

$$P = \frac{b^4(n+\cos.\zeta)\sqrt{(1+nn+2n\cos.\zeta)}}{(1+n\cos.\zeta)^3},$$

$$Q = \frac{b^4(n+\cos.\eta)\sqrt{(1+nn+2n\cos.\eta)}}{(1+n\cos.\eta)^3},$$

$$R = \frac{b^4(n+\cos.\theta)\sqrt{(1+nn+2n\cos.\theta)}}{(1+n\cos.\theta)^3},$$

hinc iam, si inter ternas applicatas p, q, r relatio supra indicata statuatur, haec arcuum comparatio obtinebitur

$$\Gamma:\zeta + \Gamma:\eta + \Gamma:\theta = \frac{nnb\sin.\zeta\sin.\eta\sin.\theta}{(1+n\cos.\zeta)(1+n\cos.\eta)(1+n\cos.\theta)}.$$

22. Relatio autem inter litteras p, q, r stabilienda ad nostros angulos traducta erat

$$r(b^4 + nn(1-nn)ppqq) = -pQ - qP,$$

cuius membrum sinistrum facta substitutione induet hanc formam

$$\frac{b^5\sin.\theta(1+2n(\cos.\zeta+\cos.\eta)+nn(1+4\cos.\zeta\cos.\eta+\cos.\zeta^2\cos.\eta^2))}{(1+n\cos.\theta)(1+n\cos.\zeta)^2(1+n\cos.\eta)^2}$$
$$+\frac{b^5\sin.\theta\,(2n^3\cos.\zeta\cos.\eta(\cos.\zeta+\cos.\eta)+n^4(\cos.\zeta^2+\cos.\eta^2-1))}{(1+n\cos.\theta)(1+n\cos.\zeta)^2(1+n\cos.\eta)^2},$$

membrum vero dextrum ad hanc formam reducitur

$$-\frac{b^5\sin.\zeta(n+\cos.\eta)\sqrt{(1+nn+2n\cos.\eta)}}{(1+n\cos.\zeta)(1+n\cos.\eta)^3} - \frac{b^5\sin.\eta(n+\cos.\zeta)\sqrt{(1+nn+2n\cos.\zeta)}}{(1+n\cos.\eta)(1+n\cos.\zeta)^3}.$$

Hic quidem utrinque per b^5 dividi potest neque tamen hinc patet, quomodo angulus θ ex binis reliquis angulis ζ et η definiri queat.

DIGRESSIO AD PARABOLAM

23. Quoniam igitur non patet, quomodo in genere ex binis angulorum ζ et η tertium determinari oporteat, hanc investigationem ad parabolam transferamus ponendo $n=1$; tum autem membrum illud sinistrum abit in

$$\frac{\sin.\theta}{1+\cos.\theta}=\operatorname{tang}.\frac{1}{2}\theta,$$

membrum autem dextrum evadit

$$-\frac{\sin.\zeta\sqrt{(2+2\cos.\eta)}}{(1+\cos.\zeta)(1+\cos.\eta)}-\frac{\sin.\eta\sqrt{(2+2\cos.\zeta)}}{(1+\cos.\zeta)(1+\cos.\eta)}=-\frac{\operatorname{tang}.\frac{1}{2}\zeta}{\cos.\frac{1}{2}\eta}-\frac{\operatorname{tang}.\frac{1}{2}\eta}{\cos.\frac{1}{2}\zeta},$$

ita ut aequatio nostra prodierit

$$\operatorname{tang}.\frac{1}{2}\theta=-\frac{\operatorname{tang}.\frac{1}{2}\zeta}{\cos.\frac{1}{2}\eta}-\frac{\operatorname{tang}.\frac{1}{2}\eta}{\cos.\frac{1}{2}\zeta}=\frac{-\sin.\frac{1}{2}\zeta-\sin.\frac{1}{2}\eta}{\cos.\frac{1}{2}\zeta\cos.\frac{1}{2}\eta}.$$

24. Quod quo clarius appareat, notetur esse

$$p=b\operatorname{tang}.\frac{1}{2}\zeta,\quad q=b\operatorname{tang}.\frac{1}{2}\eta,\quad r=b\operatorname{tang}.\frac{1}{2}\theta,$$

praeterea vero

$$P=\frac{b^4}{\cos.\frac{1}{2}\zeta},\quad Q=\frac{b^4}{\cos.\frac{1}{2}\eta},\quad R=\frac{b^4}{\cos.\frac{1}{2}\theta}.$$

Cum igitur etiam pro hoc casu prodeat $b^4R=b^6pq+PQ$, erit

$$\frac{1}{\cos.\frac{1}{2}\theta}=\frac{1+\sin.\frac{1}{2}\zeta\sin.\frac{1}{2}\eta}{\cos.\frac{1}{2}\zeta\cos.\frac{1}{2}\eta};$$

ante autem invenimus

$$\operatorname{tang}.\frac{1}{2}\theta=\frac{-\sin.\frac{1}{2}\zeta-\sin.\frac{1}{2}\eta}{\cos.\frac{1}{2}\zeta\cos.\frac{1}{2}\eta},$$

unde haec aequatio per illam divisa praebet

$$\sin.\frac{1}{2}\theta = \frac{-\sin.\frac{1}{2}\zeta - \sin.\frac{1}{2}\eta}{1+\sin.\frac{1}{2}\zeta\sin.\frac{1}{2}\eta}$$

sive

$$\sin.\frac{1}{2}\theta + \sin.\frac{1}{2}\zeta + \sin.\frac{1}{2}\eta + \sin.\frac{1}{2}\zeta\sin.\frac{1}{2}\eta\sin.\frac{1}{2}\theta = 0,$$

in qua aequatione terni anguli ζ, η, θ sunt permutabiles, quemadmodum rei natura postulat, quae proprietas in valore primo invento non tam erat manifesta.

25. Quodsi ergo terni anguli ζ, η, θ ita a se invicem pendeant, ut sit

$$\sin.\frac{1}{2}\zeta + \sin.\frac{1}{2}\eta + \sin.\frac{1}{2}\theta + \sin.\frac{1}{2}\zeta\sin.\frac{1}{2}\eta\sin.\frac{1}{2}\theta = 0,$$

tum in parabola terni arcus his angulis ζ, η, θ respondentes semper ita erunt comparati, ut sit

$$\Gamma:\zeta + \Gamma:\eta + \Gamma:\theta = b\,\mathrm{tang}.\frac{1}{2}\zeta\,\mathrm{tang}.\frac{1}{2}\eta\,\mathrm{tang}.\frac{1}{2}\theta.$$

Hinc si dati fuerint bini anguli ζ et η, tertius θ ope formulae primum inventae facillime definitur, qua erat

$$\mathrm{tang}.\frac{1}{2}\theta = \frac{-\sin.\frac{1}{2}\zeta - \sin.\frac{1}{2}\eta}{\cos.\frac{1}{2}\zeta\cos.\frac{1}{2}\eta},$$

quae expressio per meros factores ita exhiberi potest

$$\mathrm{tang}.\frac{1}{2}\theta = \frac{-2\sin.\frac{\zeta+\eta}{4}\cos.\frac{\zeta-\eta}{4}}{\cos.\frac{1}{2}\zeta\cos.\frac{1}{2}\eta};$$

unde patet, si anguli ζ et η fuerint positivi, tertium θ necessario fieri negativum sive arcum ipsi respondentem negative capi debere. Ceterum patet, si unus horum angulorum, veluti ζ, evanescat, tum fore $\sin.\frac{1}{2}\theta + \sin.\frac{1}{2}\eta = 0$ sive summam duorum reliquorum nihilo aequari sive alterum alterius fieri negativum.

PROBLEMA

26. *In quadrante elliptico* AOC (Fig. 3), *sumto pro lubitu arcu* AQ, *ab altero termino* C *abscindere arcum* CR, *qui illum arcum* AQ *superet quantitate algebraica.*

SOLUTIO

Sint huius ellipsis semiaxes ut supra $OA = a$ et $OC = c$, et cum sit arcus $CR = AC - AR$, requiritur, ut fiat $AC - AR - AQ$ quantitas algebraica. Ducantur ad axem OA perpendicula Qq et Rr, quae vocentur $Qq = q$ et $Rr = r$, quae respectu formularum supra inventarum capi debent negativa, quia arcus respondentes AQ et AR hic negative capiuntur. Cum igitur arcus $\Pi : p$ hic sit quadrans AC, erit

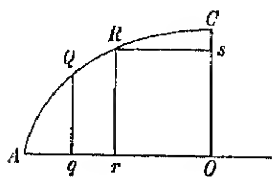

Fig. 3.

$$p = c, \quad A = c^8, \quad C = c^4(aa - 2cc), \quad E = -cc(aa - cc);$$

pro applicata quacunque z vero erit formula respondens

$$Z = c^5 \sqrt{(cc - zz)(c^4 + (aa - cc)zz)},$$

unde pro casu $z = c$ fiet $Z = 0$, quocirca pro praesenti casu, ubi $p = c$, erit $P = 0$. Deinde autem si loco q ibi scribatur $-q$, fiet

$$Q = c^5 \sqrt{(cc - qq)(c^4 + (aa - cc)qq)}.$$

27. Sumtis autem litteris q et r negativis, cum in genere invenerimus

$$r = \frac{-pQ - qP}{A - Eppqq},$$

ob $p = c$ et $P = 0$ fiet

$$-r = \frac{-cQ}{A - Eccqq} \quad \text{ideoque} \quad r = \frac{cc\sqrt{(cc - qq)(c^4 + (aa - cc)qq)}}{c^4 + (aa - cc)qq},$$

quo valore invento erit differentia arcuum $CR - AQ$ sive

$$\Pi : c - \Pi : q - \Pi : r = \frac{M}{\sqrt{A}} \cdot pqr = \frac{aa - cc}{c^3} \cdot qr;$$

quamobrem si loco r valorem inventum substituamus, habebimus

$$CR - AQ = \frac{(aa-cc)q\sqrt{(cc-qq)}(c^4+(aa-cc)qq)}{c(c^4+(aa-cc)qq)}.$$

Hic igitur quantitas q arbitrio nostro est relicta, unde arcum AQ pro lubitu assumere licet, hincque punctum R seu applicata $Rr = r$ ita est determinata, ut differentia arcuum $CR - AQ$ fiat algebraica; formulae autem inventae manifesto reducuntur ad has simpliciores

$$r = \frac{cc\sqrt{(cc-qq)}}{\sqrt{(c^4+(aa-cc)qq)}}$$

et differentia arcuum

$$CR - AQ = \frac{(aa-cc)q\sqrt{(cc-qq)}}{c\sqrt{(c^4+(aa-cc)qq)}},$$

ubi notetur esse arcum

$$AQ = \int \frac{dq\sqrt{(c^4+(aa-cc)qq)}}{c\sqrt{(cc-qq)}}.$$

28. Quoniam puncta Q et R inter se permutari possunt, siquidem est

$$CR - AQ = CQ - AR,$$

hanc permutabilitatem etiam valor pro r inventus ostendit. Sumtis enim quadratis obtinebitur ista aequatio

$$c^6 - c^4(qq+rr) - (aa-cc)qqrr = 0,$$

quae manifesto reducitur ad hanc formam concinniorem

$$(cc-qq)(cc-rr) = \frac{aaqqrr}{cc};$$

unde si statuamus $qr = uu$, ut sit $qqrr = u^4$, ex hac aequatione erit

$$qq + rr = cc - \frac{(aa-cc)}{c^4}u^4;$$

quare si $2qr = 2uu$ sive addatur sive subtrahatur, colligitur fore

$$q + r = \sqrt{cc + 2uu - \frac{(aa-cc)u^4}{c^4}}$$

et

$$q - r = \sqrt{cc - 2uu - \frac{(aa-cc)u^4}{c^4}};$$

unde sumto u pro lubitu ambae quantitates q et r simili modo exprimuntur.

Hoc modo etiam facile effici potest, ut ambo puncta Q et R congruant; facto enim $q - r = 0$ fiet $uu = -\frac{c^4 \pm ac^3}{aa - cc}$; erit ergo vel

$$uu = \frac{c^3}{a+c} \quad \text{vel} \quad uu = -\frac{c^3}{a-c};$$

tum autem erit

$$qq = \frac{c^3}{a+c} \quad \text{vel} \quad qq = -\frac{c^3}{a-c},$$

quorum valorum positivum sumi oportet. Quia autem q superare nequit c, prior tantum valor locum habere potest, quo est $qq = \frac{c^3}{a+c}$.

29. Conveniant igitur ambo haec puncta in puncto U (Fig. 4), ita ut sit applicata $Uu = \frac{c\sqrt{c}}{\sqrt{(a+c)}}$; tum vero erit arcuum differentia

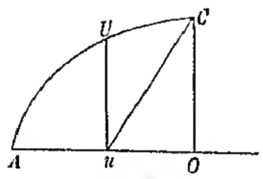

Fig. 4.

$$CU - AU = \frac{aa - cc}{a+c} = a - c,$$

ita ut haec differentia aequetur ipsi differentiae axium OA et OC. Hinc igitur erit $AO + AU = CO + CU$, ubi manifestum est, si esset $a = c$, tum punctum U in medium arcus AC incidere. Ad hoc punctum U clarius intelligendum quaeramus etiam intervallum Ou, et cum sit

$$\frac{Ou^2}{aa} + \frac{Uu^2}{cc} = 1, \quad \text{erit} \quad Ou^2 = aa - \frac{aac}{a+c} = \frac{a^3}{a+c},$$

unde patet fore $\frac{Uu}{Ou} = \frac{c\sqrt{c}}{a\sqrt{a}}$, quae ergo est tangens anguli AOU.

30. Quia in ellipsi ambo semiaxes a et c sunt permutabiles, quemadmodum arcus AQ (Fig. 3, p. 72) definitur per applicatam $Qq = q$, simili modo permutatis axibus arcus CR definietur per applicatam $Rs = Or$. Posita igitur

$Rs = s$ erit per formulam integralem arcus

$$CR = \int \frac{ds\sqrt{(a^4 - (aa-cc)ss)}}{a\sqrt{(aa-ss)}}$$

sicque erit

$$\int \frac{ds\sqrt{(a^4 - (aa-cc)ss)}}{a\sqrt{(aa-ss)}} - \int \frac{dq\sqrt{(c^4 + (aa-cc)qq)}}{c\sqrt{(cc-qq)}}$$
$$= \frac{(aa-cc)qr}{c^3} = \frac{(aa-cc)q\sqrt{(cc-qq)}}{c\sqrt{(c^4 + (aa-cc)qq)}}.$$

Videamus igitur, quomodo s se habeat respectu q; primo autem erit $\frac{ss}{aa} + \frac{rr}{cc} = 1$, unde fit

$$ss = aa - \frac{aa}{cc}rr = \frac{a^4qq}{c^4 + (aa-cc)qq},$$

consequenter

$$c^4ss + (aa-cc)qqss - a^4qq = 0;$$

unde patet permutatis litteris a et c etiam permutari q et s, uti rei natura postulat.

31. Hinc igitur colligimus istud theorema analyticum:

THEOREMA

Si capiatur

$$s = \frac{aaq}{\sqrt{(c^4 + (aa-cc)qq)}},$$

erit differentia istarum formularum integralium semper algebraica:

$$\int \frac{ds\sqrt{(a^4 - (aa-cc)ss)}}{a\sqrt{(aa-ss)}} - \int \frac{dq\sqrt{(c^4 + (aa-cc)qq)}}{c\sqrt{(cc-qq)}} = \frac{(aa-cc)q\sqrt{(cc-qq)}}{c\sqrt{(c^4 + (aa-cc)qq)}}.$$

32. Operae igitur pretium erit per evolutionem calculi hanc egregiam reductionem ostendisse. Primo igitur cum sit

$$s = \frac{aaq}{\sqrt{(c^4 + (aa-cc)qq)}},$$

erit

$$\sqrt{(aa-ss)} = \frac{ac\sqrt{(cc-qq)}}{\sqrt{(c^4+(aa-cc)qq)}}$$

et

$$\sqrt{\left(a^4-(aa-cc)ss\right)} = \frac{aacc}{\sqrt{(c^4+(aa-cc)qq)}},$$

unde fit pro prima formula integrali

$$\frac{\sqrt{(a^4-(aa-cc)ss)}}{a\sqrt{(aa-ss)}} = \frac{c}{\sqrt{(cc-qq)}}.$$

Deinde vero reperitur

$$ds = \frac{aac^4 dq}{(c^4+(aa-cc)qq)^{\frac{3}{2}}};$$

hinc igitur formularum integralium prior erit

$$\int \frac{ds\sqrt{(a^4-(aa-cc)ss)}}{a\sqrt{(aa-ss)}} = \int \frac{aac^6 dq}{c(c^4+(aa-cc)qq)^{\frac{3}{2}}\sqrt{(cc-qq)}};$$

ab hac igitur si subtrahatur altera

$$\int \frac{dq\sqrt{c^4+(aa-cc)qq)}}{c\sqrt{(cc-qq)}},$$

differentiam integrabilem esse oportet. Facta autem reductione ad communem denominatorem haec differentia fit

$$\int \frac{(aa-cc)dq(c^6-2c^4qq-(aa-cc)q^4)}{c(c^4+(aa-cc)qq)^{\frac{3}{2}}\sqrt{(cc-qq)}},$$

cuius integrale ergo esse debet

$$\frac{(aa-cc)q\sqrt{(cc-qq)}}{c\sqrt{(c^4+(aa-cc)qq)}},$$

quod tentanti mox patebit. Nullum autem est dubium, quin iste casus, si probe perpendatur, largum campum sit aperturus huiusmodi investigationes adcuratius excolendi.

33. Solutio autem istius problematis elegantius sequenti modo adornari potest. Cum sit $Qq = q$, erit $Oq = \frac{a}{c}\sqrt{(cc-qq)}$ similique modo ob $Rs = s$

erit $Os = \frac{c}{a}\sqrt{(aa-ss)}$; quare, cum inter q et s ista inventa sit aequatio $s = \frac{aaq}{\sqrt{(c^4+(aa-cc)qq)}}$, erit

$$ccss(cc-qq) = aaqq(aa-ss)$$

ideoque

$$\frac{cs}{\sqrt{(aa-ss)}} = \frac{aq}{\sqrt{(cc-qq)}} \quad \text{sive} \quad \frac{cc}{a}\cdot\frac{Rs}{Os} = \frac{aa}{c}\cdot\frac{Qq}{Oq}.$$

Hinc si duci intelligantur rectae OQ et OR et vocentur anguli $AOQ = \varphi$ et $COR = \psi$, erit $\frac{cc}{a}$ tang. $\psi = \frac{aa}{c}$ tang. φ sive hi anguli ita sunt comparati, ut sit tang. ψ : tang. $\varphi = a^3 : c^3$, sicque ex angulo φ pro lubitu assumto facile definitur angulus ψ.

34. Deinde, cum inventa sit arcuum differentia

$$CR - AQ = \frac{(aa-cc)q\sqrt{(cc-qq)}}{c\sqrt{(c^4+(aa-cc)qq)}},$$

ob $\sqrt{(c^4+(aa-cc)qq)} = \frac{aaq}{s}$ erit

$$CR - AQ = \frac{(aa-cc)s\sqrt{(cc-qq)}}{aac} = \frac{s}{c}\sqrt{(cc-qq)} - \frac{c}{aa}s\sqrt{(cc-qq)}$$
$$= \frac{s\sqrt{(cc-qq)}}{c} - \frac{q\sqrt{(aa-ss)}}{a} = qs\left(\frac{\sqrt{(cc-qq)}}{cq} - \frac{\sqrt{(aa-ss)}}{as}\right),$$

quae expressio ob

$$\text{tang.}\ \varphi = \frac{cq}{a\sqrt{(cc-qq)}} \quad \text{et} \quad \text{tang.}\ \psi = \frac{as}{c\sqrt{(aa-ss)}}$$

ad hanc formam reducitur

$$qs\left(\frac{\text{cot.}\ \varphi}{a} - \frac{\text{cot.}\ \psi}{c}\right).$$

THEOREMATA QUAEDAM ANALYTICA QUORUM DEMONSTRATIO ADHUC DESIDERATUR

Commentatio 590 indicis ENESTROEMIANI
Opuscula analytica 2, 1785, p. 76—90

1. In Analysi diophantea, quae circa proprietates numerorum [illegible] notissimum est plurima occurrere theoremata, de quorum veritate [illegible] non licet, etiamsi ea demonstratione rigida confirmare non valeamus. [illegible] Geometria autem nemo adhuc eiusmodi theoremata in medium [illegible] quorum vel veritatem vel falsitatem demonstrare non liceat. At [illegible] Analysi sublimiori iam dudum etiam eiusmodi theoremata se mihi [illegible] quorum demonstrationem nullo modo etiam nunc invenire potui, [illegible] eorum veritas nequaquam in dubium vocari videatur. Talia igitur theoremata [illegible] utique summam attentionem merentur, cum nullum plane sit dubium, [illegible] si eorum demonstrationem adhuc frustra acquisitam detexerimus, inde [illegible] momenti incrementa in Analysin sint redundatura.

2. Inter huiusmodi autem veritates analyticas merito primum [illegible] tribuo insigni illi proprietati quantitatum imaginariarum, quod, ubi [illegible] tales quantitates natura sua impossibiles occurrant, eae semper in [illegible] hac $a + b\sqrt{-1}$ comprehendi queant. Huic quidem veritati innititur [illegible] omnium aequationum algebraicarum; quippe quarum radices nisi [illegible] reales, omnes in tali formula $a + b\sqrt{-1}$ contineri perhibentur, id quod [illegible] illustris D'ALEMBERT[1]) demonstratione perquam ingeniosa confirmavit. [illegible]

1) I. D'ALEMBERT, *Recherches sur le calcul intégral*, Mém. de l'acad. d. sc. de [illegible] (1746), 1748, p. 182. A. K.

autem quoniam ex consideratione infinite parvorum est petita, haud immerito adhuc demonstratio planior ex ipsa natura imaginariorum petenda desideratur. Praeterea vero ista demonstratio tantum ad expressiones algebraicas patet, cum tamen aeque certum sit eam etiam in omnis generis quantitatibus transcendentibus locum habere, ubi ratiocinium, quo Vir celeberr. est usus, non semper adhiberi potest, id quod operae pretium erit clarius ostendisse.

3. Consideretur curva algebraica, ex quotcunque ramis fuerit composita, cuiusmodi sit ramus $FNLMH$ (Fig. 1), qui ad axem AK relatus, postquam ab F dextrorsum usque ad L processerit, hinc iterum sinistrorsum per LMH porrigatur, ita ut, si applicata KL hanc curvam in extremitate L tangat, abscissae cuilibet AP minori quam AK duplex respondeat applicata PM et PN. Unde si ponatur abscissa $AP = x$, applicata y duplicem habebit valorem ex tali aequatione quadratica

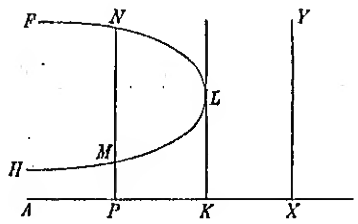

Fig. 1.

$$yy = 2py - q$$

determinandum, ita ut hinc sit altera applicata $PM = p - \sqrt{(pp - q)}$, altera vero $PN = p + \sqrt{(pp - q)}$, ubi pro indole curvae litterae p et q functiones quascunque abscissae x denotare possunt. Quamdiu igitur fuerit $pp > q$, revera gemina orietur applicata PM et PN. Dum autem abscissa x usque in K augetur, ubi fiat $pp = q$, ibi ambae applicatae in unam KL coalescent, ita ut hic applicata KL evadat curvae tangens. Quodsi ergo abscissam x ulterius augendo fiat $q > pp$, ambae applicatae evadent imaginariae. Unde intelligitur, si capiatur abscissa $AX > AK$, in hoc loco nullam prorsus dari applicatam seu rectam in hoc loco perpendicularem XY utrinque etiam in infinitum productam nusquam curvae FLH esse occursuram, id quod more loquendi in Analysi recepto idem significat ac applicatam in hoc loco X esse imaginariam; unde simul notio imaginariorum, uti in Analysi adpellantur, clarius intelligitur. Cum enim haec applicata XY curvae nusquam occurrat, etiamsi a puncto X, ubi est $= 0$, tam sursum usque in infinitum positivum quam deorsum usque in infinitum negativum continuetur, evidens est eius valorem inventum neque esse $= 0$ neque maiorem quam 0 neque minorem quam 0, qua conditione definitio ipsa quantitatum imaginariarum continetur. Quodsi ergo pro hoc loco sumamus fieri $q = pp + rr$, gemina expressio applicatae evadet

$$y = p \pm r\sqrt{-1}.$$

4. Hic igitur quaeritur, num hinc certo in genere concludi possit, quotiescunque imaginaria occurrant, ea semper huiusmodi formula $p + r\sqrt{-1}$ exprimi posse. Primo enim haec demonstratio tantum ex ramo FLH est petita, dum tota curva aequatione inter x et y contenta fortasse plures insuper alios ramos involvat, quos in hoc negotio penitus negligere fortasse non licet. Hanc autem obiectionem Vir excell. utique ipse praevidit, dum hoc ratiocinium tantum ad portiunculam curvae infinite parvam NLM extendit, ubi ulteriorem ramorum extensionem tuto negligere liceat, quod autem non adeo in aprico situm videtur, ut non planiorem demonstrationem a tali conceptu immunem merito desiderare queamus. Tum vero etiam hinc plus non sequeretur, quam applicatas XY extremae KL infinite propinquas tali formula $p \pm r\sqrt{-1}$ exprimi posse, ac non immerito dubitare liceret, an pro intervallis maioribus KX etiam applicatae tali formula comprehendi queant et annon reliquae curvae partes hactenus neglectae indolem imaginarii in his locis penitus immutare valeant.

5. Praeterea vero ista consideratio tantum ad aequationes et curvas algebraicas est accommodata, in quibus utique alii rami non dantur, nisi qui vel in se redeant vel utrinque in infinitum excurrant, ita ut circa terminum L portio curvae hic semper binas portiones LM et LN exhibeat, unde aequatio illa quadratica $yy = 2py - q$ est nata, cui tota demonstratio innititur. At vero inter curvas transcendentes eiusmodi rami occurrunt, qui neque utrinque in infinitum protenduntur neque in se redeunt, sed subito in quopiam puncto terminantur. Talem casum praebet curva transcendens hac aequatione contenta

$$y = a + \frac{bx}{l(c-x)},$$

ex qua sequitur singulis abscissis unicam tantum applicatam respondere. Posito enim $x = 0$ fit $y = a$; ac si abscissa x continuo augeatur usque ad valorem $x = c$, perpetuo unica dabitur applicata; sumta vero abscissa $x = c$ ob $l(c-x) = -\infty$ fiet applicata in hoc loco $y = a$. Statim autem atque abscissa x ultra c augetur, applicata subito fiet imaginaria, propterea quod logarithmi quantitatum negativarum certo sunt imaginarii; quare sumta abscissa $x > c$ applicata y, etiamsi utrinque in infinitum producatur, curvae tamen nostrae nusquam occurret. Hoc autem casu ratio supra allegata et naturae aequationis quadraticae innixa penitus cessat, ita ut hic merito

dubitare possimus, an ista applicata imaginaria etiam in formula $p + q\sqrt{-1}$ comprehendi queat. Saltem hic agnoscere debemus istud theorema alia demonstratione indigere ideoque maxime optandum esse, ut talis aequatio immediate ex ipsa natura imaginariorum derivetur.

6. Ante autem quam hoc argumentum deseram, ostendisse iuvabit, quomodo omnia plane imaginaria singulari prorsus ratione per circulum repraesentari possint. Ex puncto A (Fig. 2) pro principio axis AB assumto erigatur perpendiculum $AC = a$; centro C radio $CM = c$ describatur circulus ac posita abscissa quacunque $AP = x$ eique respondente applicata $PM = y$ erit

$$y = AC + QM = AC + \sqrt{(CM^2 - CQ^2)} = a + \sqrt{(cc - xx)},$$

ita ut eius valor semper sit realis, quamdiu abscissa x minor capitur quam radius c; simulac vero abscissa x radium c superat, veluti si sumatur $x = AX$, tum applicata XY certe erit imaginaria. At vero, quanquam ob hanc ipsam causam applicata exhiberi nequit, tamen determinatum habet valorem imaginarium (iam enim evictum est notionem determinati notioni imaginarii non adversari). Quoniam enim ponitur $x > c$, statuatur $xx = cc + bb$, ut fiat $\sqrt{(cc - xx)} = b\sqrt{-1}$ ideoque applicata ista imaginaria $XY = a + b\sqrt{-1}$. Quare cum formula $a + b\sqrt{-1}$ omnes plane quantitates imaginarias contineat,

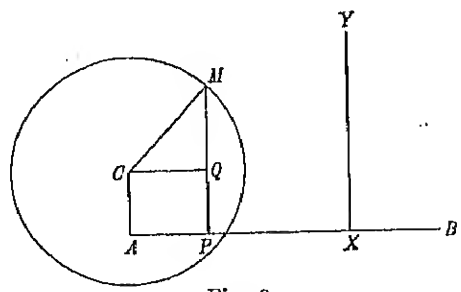

Fig. 2.

eas per huiusmodi applicatam determinatam XY ad circulum quendam pertinentem repraesentare licebit. Posito scilicet perpendiculo $AC = a$ centro C radio pro arbitrio assumto c describatur circulus ac sumatur abscissa $AX = \sqrt{(bb + cc)}$; tum enim applicata imaginaria XY istam formulam $a + b\sqrt{-1}$ exhibebit sicque mirabili quodam modo omnes adeo formulas imaginarias quasi geometrice construere licebit.

7. Operae pretium erit hoc exemplo quodam declarasse. Quaeramus scilicet arcum circuli, cuius sinus duplo maior sit sinu toto, qui ergo certe erit imaginarius. Posito ergo sinu toto $= 1$ integrari debet formula $\frac{dx}{\sqrt{(1 - xx)}}$, ita ut integrale evanescat posito $x = 0$; tum vero sumi debebit $x = 2$ et valor

integralis dabit ipsum arcum. Hunc in finem formulae differentiali $\frac{dx}{\sqrt{(1-xx)}}$ tribuamus hanc formam $\frac{dx\sqrt{-1}}{\sqrt{(xx-1)}}$; constat autem esse

$$\int \frac{dx}{\sqrt{(xx-1)}} = l\frac{x+\sqrt{(xx-1)}}{\sqrt{-1}},$$

unde posito $x=2$ arcus quaesitus erit

$$=\sqrt{-1}\, l\frac{2+\sqrt{3}}{\sqrt{-1}} = \sqrt{-1}\, l(2+\sqrt{3}) - \sqrt{-1}\, l\sqrt{-1}.$$

Novimus autem huius postremi membri valorem esse $\frac{\pi}{2}$, unde arcus circuli, cuius sinus $=2$, erit $\frac{\pi}{2}+\sqrt{-1}\, l(2+\sqrt{3})$. Quamobrem ut huic arcui imaginario aequalem applicatam XY exhibeamus, in nostra figura capiatur intervallum $AC=\frac{\pi}{2}$ ac descripto circulo radii $CM=c=1$, quia c arbitrio nostro relinquitur, posito brevitatis gratia $l(2+\sqrt{3})=b$ capiatur abscissa $AX=\sqrt{(1+bb)}$ atque applicata imaginaria XY aequalis erit ipsi arcui quaesito pariter imaginario, id quod eo magis notatu dignum videtur, quod iste arcus est imaginarium transcendens.

8. Primum igitur theorema analyticum, cuius demonstratio planior vel saltem magis directa desideratur, siquidem eius veritas quibusdam iam satis evicta videatur, hoc modo proponatur:

THEOREMA 1

Omnes plane quantitates imaginariae, quaecunque in calculo analytico occurrere possunt, ad hanc formam simplicissimam $a+b\sqrt{-1}$ ita revocari possunt, ut litterae a et b quantitates reales denotent.

Eius igitur demonstrationem sagacissimis analystis imprimis commendare non dubito.

9. Sequentia duo theoremata rectificationem linearum curvarum respiciunt ideoque ad geometriam sublimiorem sunt referenda. Cum enim iam pridem a celeb. HERMANNO[1]) methodus geometrica sit reperta innumerabiles curvas

1) IAC. HERMANN, *Solutio propria duorum problematum geometricorum in Actis Erudit. 1719 Mens. Aug. a se propositorum*, Acta erud. 1723, p. 171. A. K.

algebraicas inveniendi, quae vel sint rectificabiles vel quarum rectificatio a data quacunque quadratura pendeat (quam methodum deinceps ad analysin puram transtuli et plurimum locupletavi, ita ut peculiarem speciem analyseos infinitorum constituere videatur), inde utique infinitae curvae algebraice exhiberi possunt, quarum rectificatio a quadratura circuli pendeat. Omnes autem excepto circulo ita comparatae deprehenduntur, ut earum arcus aggregato cuipiam ex quantitate algebraica et arcu circulari aequentur, quantitatem autem illam algebraicam nullo modo ad nihilum redigere liceat; unde sequens theorema tanquam verum proponere non dubito, etiamsi eius demonstrationem exhibere nondum potuerim.[1])

THEOREMA 2

Praeter circulum nulla datur curva algebraica, cuius singuli arcus per arcus circulares simpliciter exprimi queant.

10. Hoc theorema igitur eo redit, ut demonstretur nullam aequationem algebraicam inter binas coordinatas orthogonales x et y exhiberi posse, ut formula integralis $\int \sqrt{(dx^2+dy^2)}$ aequetur arcui cuipiam circulari, cuius sinus vel cosinus sit functio quaepiam [algebraica] ipsarum x et y, solo casu excepto, quo aequatio inter x et y circulum indicat. Quod quo clarius intelligatur, denotet φ angulum seu arcum quemcunque indefinitum in circulo, cuius radius $=1$, ac ponatur

$$\int \sqrt{(dx^2+dy^2)} = a\varphi$$

ideoque

$$dx^2+dy^2 = aad\varphi^2$$

fiatque

$$dx = apd\varphi \quad \text{et} \quad dy = aqd\varphi$$

atque necesse est, ut sit $pp+qq=1$. Praeterea vero ambas formulas $apd\varphi$ et $aqd\varphi$ ita integrabiles esse oportet, ut earum integralia per solos sinus vel cosinus anguli φ exprimi queant, quod dico nullo modo fieri posse, nisi curva fuerit ipse circulus.

1) Hoc theorema falsum esse postea EULERUS ipse invenit; vide infra Commentationem 783 (indicis ENESTROEMIANI). A. K.

11. His autem conditionibus manifesto satisfiet, si capiatur

$$p = \sin.(n\varphi + \alpha) \quad \text{et} \quad q = \cos.(n\varphi + \alpha)$$

denotante α angulum quemcunque constantem, n vero numerum rationalem quemcunque; tum enim utique erit $pp + qq = 1$, et cum sit $dx = ad\varphi \sin.(n\varphi + \alpha)$ et $dy = ad\varphi \cos.(n\varphi + \alpha)$, hinc integrando elicitur

$$x = b - \frac{a}{n}\cos.(n\varphi + \alpha) \quad \text{et} \quad y = c + \frac{a}{n}\sin.(n\varphi + \alpha),$$

quae formulae ob litteras α et n arbitrarias innumeras curvas involvere videntur. Verum cum inde fiat

$$b - x = \frac{a}{n}\cos.(n\varphi + \alpha) \quad \text{et} \quad y - c = \frac{a}{n}\sin.(n\varphi + \alpha),$$

semper erit

$$(b-x)^2 + (y-c)^2 = \frac{aa}{nn},$$

quae aequatio manifesto semper est pro circulo. Demonstrandum igitur est pro conditionibus ante praescriptis loco litterarum p et q alios valores accipi non posse, qui iis satisfaciant.

12. Cum autem nullum vestigium appareat ad talem demonstrationem perveniendi, videamus, an per demonstrationem ad absurdum quicquam lucrari possit. Assumamus igitur praeter circulum aliam dari curvam algebraicam, cuius omnes arcus per arcus circulares metiri liceat. Sit igitur $AYym$ (Fig. 3) talis curva algebraica, cuius quilibet arcus AY ab initio A captus aequetur arcui cuipiam circulari, cuius sinus sit functio quaecunque algebraica abscissae AX, ac simili modo alius arcus quicunque Ay etiam aequabitur arcui circulari, cuius sinus erit similis functio abscissae Ax; hincque manifestum est etiam differentiae horum arcuum Yy aequalem arcum circularem assignari posse, ita ut huius curvae omnes plane portiones Yy per simplices arcus circulares exprimi queant, sicque demonstrari oportebit talem curvam algebraicam nullo prorsus modo exhiberi posse.

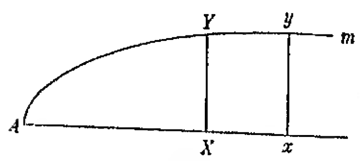

Fig. 3.

13. Primo hic autem observo, si daretur talis curva, eam certe non in infinitum extendi posse, id quod ita ostendo. Sit $Abcde$ etc. (Fig. 4) talis curva

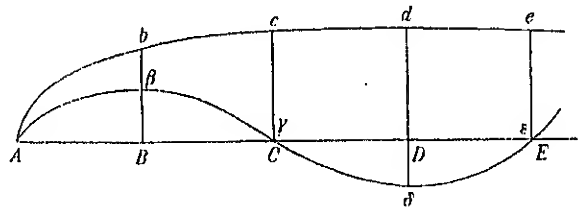

Fig. 4.

cum axe $ABCDE$ in infinitum excurrens in eaque accipiantur portiones aequales Ab, bc, cd, de etc., quarum mensura sit quadrans circuli, atque in applicatis Bb, Cc, Dd etc. abscindantur portiones $B\beta$, $C\gamma$, $D\delta$ etc., quae sint sinibus arcuum Ab, Ac, Ad etc. aequales, id quod etiam in singulis applicatis intermediis fieri intelligatur; ac manifestum est singula haec puncta β, γ, δ etc. geometrice seu algebraice assignari posse, ita ut curva per omnia haec puncta ducta $A\beta\gamma\delta\varepsilon$ etc. futura esset algebraica. Quoniam vero ea habebit infinitas portiones alternatim supra et infra axem existentes, ea ab axe ipso in infinitis punctis intersecaretur, id quod in nulla curva algebraica locum habere potest. Unde luculenter sequitur talem curvam $Abcd$ etc. in infinitum extensam certe non dari posse; atque hinc iam est evictum, si darentur praeter circulum eiusmodi curvae algebraicae, quarum singulae portiones per arcus circulares mensurari queant, necessario eas in se redeuntes esse debere; tum enim absurditas modo ostensa cessare posset, ita ut simili modo nihil absurdi inde inferri possit.

14. Sit igitur $ABPQRS$ (Fig. 5) talis curva algebraica in se rediens, cuius omnes plane portiones per arcus circulares metiri liceat, quae tamen non sit circulus; tum sumta quacunque portione AB a quovis alio puncto P abscindi poterit portio PQ illi aequalis, quae tamen illi maxime erit dissimilis, quandoquidem curvamen seu radius osculi maxime differre potest in his portionibus, quales sunt AB, PQ, RS etc. Quanquam autem in hoc equidem nullam contradictionem ostendere possum, tamen demonstrari potest, si unica talis curva daretur, ex ea infinitas alias inter se diversas geometrice construi posse. Tum vero ex qualibet earum porro simili modo infinitas alias

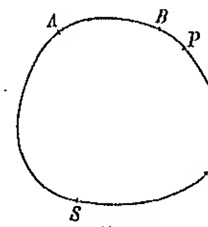

Fig. 5.

ex earumque denuo qualibet infinitas alias sicque in infinitum, ita ut multitudo talium curvarum satisfacientium foret non solum numerus infinitus, sed adeo potestas infinitesima infiniti. Quare, cum adhuc nullo modo talis curva reperiri potuerit, nonne hinc iure concludere licebit nullas plane dari huiusmodi curvas algebraicas?

15. Ad hoc autem demonstrandum insignes illae proprietates, quas Vir celeberr. IOANNES BERNOULLI[1]) de motu reptorio et curvis aeque amplis in lucem produxit, summo cum successu in usum vocari poterunt. Fundamentum autem huius eximiae methodi in hoc consistit. Si habeantur duae curvae utcunque diversae aym et $a'y'm'$ (Fig. 6 et 7) in iisque capiantur arcus ay et $a'y'$ aeque ampli, ita ut ductis ad puncta y et y' normalibus yr et $y'r'$, quae axibus ab et $a'b'$ in r et r' occurrant, qui ipsi ad curvas normales supponuntur

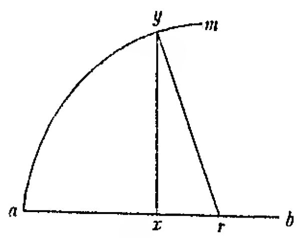

Fig. 6.

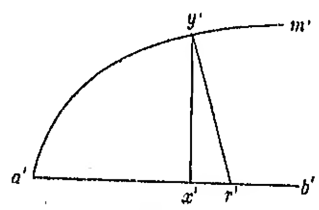

Fig. 7.

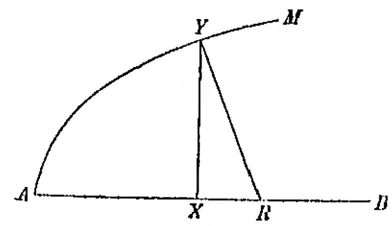

Fig. 8.

in a et a', anguli ary et $a'r'y'$ fiant inter se aequales, ex quo hi arcus ay et $a'y'$ aeque ampli sunt appellati, quibus positis si hinc nova curva AYM (Fig. 8) ita construatur, ut sumta abscissa $AX = m \cdot ax + n \cdot a'x'$ constituatur applicata $XY = m \cdot xy + n \cdot x'y'$, tum etiam huius novae curvae AM arcus AY erit $= m \cdot ay + n \cdot a'y'$. Quodsi enim pro curvis datis ponamus abscissas $ax = x$ et $a'x' = x'$, applicatas vero $xy = y$ et $x'y' = y'$, erit subnormalis $xr = \frac{ydy}{dx}$ et $x'r' = \frac{y'dy'}{dx'}$ hincque tang. $ary = \frac{dx}{dy}$ et tang. $a'r'y' = \frac{dx'}{dy'}$. Quare cum hi anguli sint aequales, posito $\frac{dy}{dx} = p$ seu $dy = pdx$ erit etiam $\frac{dy'}{dx'} = p$ sive $dy' = pdx'$. Hinc igitur colligitur arcus $ay = \int dx\sqrt{(1+pp)}$ et arcus $a'y' = \int dx'\sqrt{(1+pp)}$. Iam in curva inde constructa AY erit abscissa

1) IOH. BERNOULLI, *Motus reptorius eiusque insignis usus pro lineis curvis in unam omnibus aequalem colligendis vel a se mutuo subtrahendis; atque hinc deducta problematis de transformatione curvarum in Diario Gallico Paris. 12. Febr. 1702 propositi genuina solutio.* Acta erud. 1705, p. 347; *Opera omnia* t. 1, p. 408. A. K.

$AX = X = mx + nx'$, applicata vero $XY = Y = my + ny'$ hincque

$$dX = mdx + ndx' \quad \text{et} \quad dY = mdy + ndy' = p(mdx + ndx')$$

ideoque erit $dY = pdX$ et arcus AY aeque amplus erit ac duo praecedentes ay et $a'y'$; hinc ergo huius novae curvae arcus erit

$$AY = \int dX\sqrt{(1+pp)} = m\int dx\sqrt{(1+pp)} + n\int dx'\sqrt{(1+pp)},$$

unde manifestum est fore arcum $AY = m \cdot ay + n \cdot a'y'$.

16. Hoc iam fundamento stabilito si ambae curvae ay et $a'y'$ ita fuerint comparatae, ut arcus ay et $a'y'$ per arcus circulares mensurari queant, tum etiam curvae inde descriptae arcus AY etiam per arcum circularem mensurabitur, si modo litterae m et n denotent numeros rationales quoscunque. Ex quo iam intelligitur ex illis curvis datis ay et $a'y'$ innumerabiles curvas AY eiusdem proprietatis construi posse. Hic autem observandum est, si ambae curvae datae ay et $a'y'$ fuerint circuli, curvam illam descriptam AY fore quoque circulum, cuius radius $RA = RY$ erit $= m \cdot ra + n \cdot r'a'$, ita ut hoc solo casu nulla nova curva resultet, id quod per se est perspicuum. Statim autem ac vel altera earum curvarum ay et $a'y'$ vel etiam ambae non fuerint circuli, tum quoque curva descripta AY certe non erit circulus atque adeo in infinitum variari poterit, prouti numeris m et n alii atque alii valores tribuantur.

17. Hinc ergo si pro curva ay accipiatur curva illa supra memorata, cuius scilicet singulos arcus per circulares mensurare posse assumimus, eamque a puncto quocunque A incipientem, pro altera autem $a'y'$ circulum quemcunque, constructio modo tradita nobis suppeditabit innumerabiles curvas AY eadem indole praeditas, ut arcui AY aequalis arcus circularis assignari queat. Tum vero etiam sumta curva ay aequali ramo figurae illius a puncto A extenso, pro curva vero $a'y'$ alius quicunque eiusdem curvae ramus ab alio puncto P protensus hinc etiam innumerabiles aliae novae curvae AY describi poterunt, quae utique omnes quoque erunt algebraicae; unde manifestum est, si harum novarum curvarum rami in locum alterius curvae datae ay vel etiam utriusque substituantur, tum hoc modo infinita alia curvarum genera construi posse, quam multiplicationem adeo in infinitum augere licebit. Quare, cum nulla

adhuc ciusmodi curva a circulo diversa erui potuerit, maxime verisimile est ac fortasse tanquam rigide demonstratum spectari potest nullam prorsus in rerum natura dari huiusmodi curvam algebraicam a circulo diversam.

18. Quod hactenus de circulo est allatum, etiam ad logarithmos extendi potest, quippe quos cum arcubus circularibus imaginariis comparare licet, unde sequens theorema geometris tanquam aeque certum et memoratu dignum ac praecedens commendare sustineo.

THEOREMA 3

Nulla prorsus datur curva algebraica, cuius singuli arcus simpliciter per logarithmos exprimi queant.

Ita ut hoc theorema nullam prorsus exceptionem quemadmodum praecedens postulet.

19. Notum est rectificationem parabolae a logarithmis pendere, verum singuli eius arcus non per simplices logarithmos, sed per aggregatum ex logarithmo et quapiam quantitate algebraica exprimuntur, ita ut hinc nulla exceptio theoremati inferatur. Hic autem primo observandum est ut ante, si talis daretur curva algebraica AYy (Fig. 3, p. 84), cuius omnes arcus in puncto A terminati per logarithmos assignari possent, ut verbi gratia esset $AY = alP$ et $Ay = alp$, ita ut P et p essent certae functiones algebraicae ambarum coordinatarum AX, XY et Ax, xy, tum etiam differentiam horum arcuum logarithmo exprimi posse, quandoquidem foret $Yy = al\frac{p}{P}$. Hinc ergo posita abscissa $AX = x$ et applicata $XY = y$ demonstrandum est nullam dari aequationem algebraicam inter x et y, ut inde fiat

$$\int \sqrt{(dx^2 + dy^2)} = alv$$

denotante v functionem quampiam algebraicam ipsarum x et y; unde si ponamus

$$dx = \frac{apdv}{v} \quad \text{et} \quad dy = \frac{aqdv}{v},$$

necesse est, ut fiat $pp + qq = 1$. Praeterea vero requiritur, ut ambae formulae $\int\frac{pdv}{v}$ et $\int\frac{qdv}{v}$ fiant algebraice integrabiles, cuius ergo impossibilitatem demonstrari oportet.

20. Quemadmodum mihi pro praecedente theoremate licuit ostendere nullam dari curvam in infinitum extensam illi satisfacientem, ita hic simili modo ostendi potest nullam dari curvam in se redeuntem algebraicam, quae huic theoremati conveniat. Sit enim curva $AYBDA$ (Fig. 9) curva in se rediens, cuius omnes arcus AY per logarithmos exhiberi queant, ita ut in applicata XY, si opus est, producta algebraice assignari possit punctum Z, ut arcus AY fiat $= \log. XZ$; tum ergo, quia curva in se est rediens et arcui $AYBDAY$ eaedem coordinatae AX et XY conveniunt, aliud quoque dabitur punctum Z, cuius logarithmus huic arcui aequetur. Ac si circumferentia totius curvae ponatur $= c$, infinita talia spatia XZ, XZ', XZ'', XZ''' etc. assignari poterunt, quorum logarithmi aequentur arcubus AY, $AY + c$, $AY + 2c$, $AY + 3c$ et in genere $AY + nc$ denotante n numerum integrum quemcunque tam negativum quam positivum; atque quia omnia haec puncta simili formula algebraica continebuntur, omnia quoque in eadem curva algebraica existerent, quae ergo a singulis applicatis XY productis in infinitis punctis secaretur, id quod naturae curvarum algebraicarum adversatur.

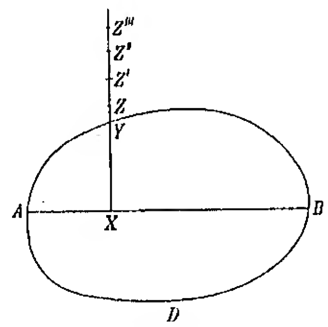

Fig. 9.

21. Quodsi ergo daretur talis curva, cuius singulos arcus logarithmis metiri liceret, ea certe in infinitum excurreret. Iam vero ex unica tali curva ope propositionis fundamentalis circa curvas aeque amplas supra allatae pari modo, quo ibi processimus, infinities-infinita nova genera talium curvarum exhiberi possent; unde, cum nulla adhuc talis curva erui potuerit, si non prorsus certum, saltem maxime verisimile est nullas plane dari eiusmodi curvas algebraicas.

22. Ceterum si modo theorema secundum firmiter fuerit demonstratum, etiam huius demonstratio pro confecta esset habenda. Cum enim elementum arcus circuli, cuius radius $= a$ et sinus $= x$, sit $\frac{adx}{\sqrt{(aa - xx)}}$, si radium ita imaginarium concipiamus, ut sit $a = c\sqrt{-1}$, elementum arcus fiet

$$\frac{cdx\sqrt{-1}}{\sqrt{(-cc - xx)}} = -\frac{cdx}{\sqrt{(cc + xx)}},$$

quod ergo erit reale, etiamsi radius circuli sit imaginarius, eiusque adeo integrale erit

$$cl\frac{\sqrt{(cc+xx)}-x}{c},$$

ubi maxime mirum videri potest, quod arcus circuli imaginarii nihilo minus sint reales et quidem per logarithmos assignabiles. Atque hinc iam tuto concludere poterimus, quemadmodum praeter circulum nullae aliae dantur lineae curvae, cuius singulos arcus per circulares metiri liceat, ita etiam praeter circulum imaginarium nullas dari curvas algebraicas, quarum singulos arcus per logarithmos metiri liceat. Quoniam autem circulus imaginarius plane existere nequit, prorsus nullae curvae algebraicae exhiberi posse sunt censendae, quarum singulos arcus per logarithmos exprimere liceat.

DE MIRIS PROPRIETATIBUS CURVAE ELASTICAE SUB AEQUATIONE $y = \int \frac{xxdx}{\sqrt{(1-x^4)}}$ CONTENTAE

Commentatio 605 indicis ENESTROEMIANI
Acta academiae scientiarum Petropolitanae 1782: II (1786), p. 34—61

1. Sit EGF (Fig. 1) lamina elastica, quae ope funiculi terminis E et F alligati incurvetur in curvam elasticam EGF; tum vero, si funiculus eo usque constringatur, donec anguli in E et F fiant recti, ea curva elastica oritur, quae vocari solet rectangula et in aequatione contenta

$$y = \int \frac{xxdx}{\sqrt{(1-x^4)}},$$

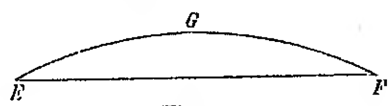

Fig. 1.

cuius nonnullas proprietates prorsus singulares et admirandas hic sum commemoraturus.

2. Sit igitur CAC' (Fig. 2) talis curva elastica rectae CC', quae funiculum refert, utrinque normaliter insistens; et evidens est rectam AD ad punctum medium D inter utrumque terminum C et C' perpendiculariter ductam fore

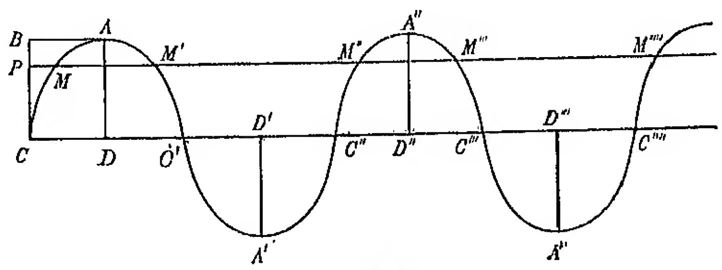

Fig. 2.

curvae diametrum et punctum A eius quasi verticem referre. Tum vero si ex C ad rectam CC' erigatur perpendiculum CB, quod tanquam axem hic spectabimus, in eoque capiamus abscissam $CP=x$ et vocemus applicatam $PM=y$, posita altitudine $AD=CB=1$ erit, uti constat,

$$dy=\frac{xxdx}{\sqrt{(1-x^4)}};$$

unde si arcus curvae CM ponatur $=s$, fiet

$$ds=\frac{dx}{\sqrt{(1-x^4)}}$$

atque tam ex natura rei quam ex hac aequatione intelligere licet totam hanc curvam constare ex infinitis portionibus CAC', $C'A'C''$, $C''A''C'''$ etc. inter se similibus et aequalibus super recta CC''' utrinque in infinitum producta constitutis, unde etiam tota haec curva infinitas habebit diametros AD, $A'D'$, $A''D''$ etc. totidemque vertices A, A', A'', A''' etc. tam dextrorsum quam sinistrorsum. Puncta autem C, C', C'', C''' etc., quoniam circa eorum singula curva similiter alternatim protenditur, centra vocari poterunt. Quemadmodum autem singularum harum portionum altitudines AD, $A'D'$, $A''D''$ etc. unitate designamus, ponamus semilatitudinem cuiusque portionis $CD=AB=a$, ipsos vero arcus $CA=C'A=C'A'=$ etc. $=c$ et, quomodo hae binae quantitates a et c se ad unitatem seu altitudinem AD habeant, deinceps accuratius investigabimus.

3. His de quantitatibus ad hanc curvam pertinentibus notatis quantitates variabiles $PM=y$ et $CM=s$ ad abscissam $CP=x$ referamus, unde statim patet tam y quam s fore functiones infinitiformes eiusdem abscissae $CP=x$. Cum enim applicata PM utrinque in infinitum producta curvam secet in infinitis punctis M, M', M'', M''' etc., applicata y infinitos recipiet valores, scilicet PM, PM', PM'', PM''', PM'''' etc., qui ex principali $PM=y$ et quantitate constante $AB=CD=a$ erunt

$$PM=y,\quad PM'=2a-y,\quad PM''=4a+y,\quad PM'''=6a-y,$$
$$PM''''=8a+y,\quad PM'''''=10a-y\quad \text{etc.},$$

qui omnes valores in his generalibus formis continentur

$$4ia+y \quad \text{et} \quad (4i+2)a-y,$$

ubi littera i omnes numeros integros tam positivos quam negativos denotare potest. Simili modo eidem abscissae $CP=x$ respondebunt infiniti arcus curvae, qui erunt

$CM=s,\quad CAM'=2c-s,\quad CAA'M''=4c+s,\quad CAA'A''M'''=6c-s$ etc.,

qui omnes etiam in his geminis formulis continentur

$$4ic+s \quad \text{et} \quad (4i+2)c-s$$

sumendo pro i successive omnes numeros tam positivos quam negativos.

4. Sufficiet igitur solam huius curvae portionem CMA (Fig. 3) considerasse, quoniam reliquae omnes ei sunt aequales, pro qua posuimus $CB=AD=1$, $AB=CD=a$ et arcum $CMA=c$. Tum vero pro puncto indefinito M si vocentur coordinatae $CP=x$, $PM=y$ et arcus $CM=s$, erit

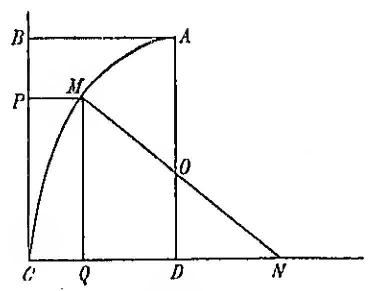

Fig. 3.

$$dy=\frac{xxdx}{\sqrt{(1-x^4)}} \quad \text{et} \quad ds=\frac{dx}{\sqrt{(1-x^4)}}.$$

His positis ad curvam in M ducamus normalem MN basi CD productae occurrentem in N. Hinc si ducatur ad basin perpendiculum $MQ=x$, ob $CQ=y$ erit intervallum

$$QN=\frac{xdx}{dy}=\frac{\sqrt{(1-x^4)}}{x}$$

et ipsa normalis

$$MN=\frac{xds}{dy}=\frac{1}{x},$$

ita ut rectangulum $MQ\cdot MN$ sit $=1=AD^2$. Hinc si vocetur angulus $CNM=\varphi$, qui metitur amplitudinem arcus CM, erit

$$\sin.\varphi=xx,\quad \cos.\varphi=\sqrt{(1-x^4)} \quad \text{et} \quad \operatorname{tang}.\varphi=\frac{xx}{\sqrt{(1-x^4)}}.$$

5. Quaeramus nunc etiam radium osculi curvae in puncto M, qui sit MO; hunc in finem faciamus

$$\frac{dy}{dx}=p=\frac{xx}{\sqrt{(1-x^4)}},$$

unde fit

$$\sqrt{(1+pp)} = \frac{1}{\sqrt{(1-x^4)}};$$

hinc porro fiat

$$\frac{p}{\sqrt{(1+pp)}} = xx = q$$

eritque, uti constat, radius osculi

$$= \frac{dx}{dq} = \frac{1}{2x};$$

sicque erit

$$MO = \frac{1}{2x}$$

ideoque

$$MO = \frac{1}{2} MN,$$

ita ut centrum curvaturae cadat in punctum medium normalis MN; ex quo patet radium osculi MO reciproce esse proportionalem intervallo $MQ = x$, quae est proprietas, quam natura elasticae postulat. Cum enim vis laminam in puncto C tendens directionem habeat CN, eius momentum respectu puncti M erit vi multiplicatae per $QM = x$ aequale, cui per naturam elasticitatis radius osculi in M reciproce debet esse proportionalis. Manifestum igitur est radium osculi in ipso puncto C esse infinitum, in altero autem termino $A = \frac{1}{2} = \frac{1}{2} AD$ sicque in hoc puncto A curvatura erit maxima.

6. Nunc etiam videamus, quomodo ex data abscissa $CP = x$ tam applicata $PM = y$ quam ipse arcus $CM = s$ proxime per series infinitas exprimi queat, id quod duplici modo praestari potest. Prior maxime obvius in eo consistit, ut formula

$$\frac{1}{\sqrt{(1-x^4)}} = (1-x^4)^{-\frac{1}{2}}$$

in seriem resolvatur, quae erit

$$1 + \frac{1}{2}x^4 + \frac{1}{2}\cdot\frac{3}{4}x^8 + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}x^{12} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}x^{16} + \text{etc.},$$

unde per integrationem colligitur

$$PM = y = \frac{1}{3}x^3 + \frac{1}{2}\cdot\frac{1}{7}x^7 + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{11}x^{11} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{15}x^{15} + \text{etc.},$$

tum vero etiam arcus

$$CM = s = x + \frac{1}{2}\cdot\frac{1}{5}x^5 + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{9}x^9 + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{13}x^{13} + \text{etc.}$$

Hinc igitur patet, si abscissa x fuerit valde parva, tum fore proxime $y=\frac{1}{3}x^3$ et $s=x$. Verum si capiamus $x=1$, per series ambae quantitates a et c ita exprimentur, ut sit

$$a = \frac{1}{3} + \frac{1}{2}\cdot\frac{1}{7} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{11} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{15} + \text{etc.}$$

$$c = 1 + \frac{1}{2}\cdot\frac{1}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{9} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{13} + \text{etc.}$$

Hae autem series nimis lente convergunt, quam ut inde valores litterarum a et c satis exacte definiri queant.

7. Alter modus non adeo obvius in eo consistit, ut statuatur

$$y = \int\frac{xxdx}{\sqrt{(1-x^4)}} = u\sqrt{(1-x^4)};$$

sumtis igitur differentialibus erit

$$xxdx = du(1-x^4) - 2ux^3dx$$

sive

$$\frac{du}{dx}(1-x^4) - 2ux^3 - xx = 0.$$

Fingatur nunc ista series

$$u = \alpha x^3 + \beta x^7 + \gamma x^{11} + \delta x^{15} + \varepsilon x^{19} + \text{etc.},$$

quandoquidem iam novimus, si x fuerit valde parvum, fieri debere $y=\frac{1}{3}x^3$ ideoque etiam $u=\frac{1}{3}x^3$; deinde ex forma aequationis manifestum est in serie

exponentes ipsius x continuo quaternario crescere debere. Hac igitur serie substituta fiat sequens evolutio

$$\begin{aligned}
\frac{du}{dx} &= 3\alpha xx + 7\beta x^6 + 11\gamma x^{10} + 15\delta x^{14} + 19\varepsilon x^{18} + \text{etc.}\\
-\frac{x^4 du}{dx} &= \qquad -3\alpha x^6 - 7\beta x^{10} - 11\gamma x^{14} - 15\delta x^{18} - \text{etc.}\\
-2ux^3 &= \qquad -2\alpha x^6 - 2\beta x^{10} - 2\gamma x^{14} - 2\delta x^{18} - \text{etc.}\\
-xx &= -xx
\end{aligned}$$

Singulis igitur membris ad nihilum redactis fiet

$$\alpha = \frac{1}{3}, \quad \beta = \frac{1 \cdot 5}{3 \cdot 7}, \quad \gamma = \frac{1 \cdot 5 \cdot 9}{3 \cdot 7 \cdot 11}, \quad \delta = \frac{1 \cdot 5 \cdot 9 \cdot 13}{3 \cdot 7 \cdot 11 \cdot 15} \quad \text{etc.},$$

quamobrem habebimus

$$y = \left(\frac{1}{3}x^3 + \frac{1 \cdot 5}{3 \cdot 7}x^7 + \frac{1 \cdot 5 \cdot 9}{3 \cdot 7 \cdot 11}x^{11} + \frac{1 \cdot 5 \cdot 9 \cdot 13}{3 \cdot 7 \cdot 11 \cdot 15}x^{15} + \text{etc.}\right)\sqrt{(1 - x^4)}.$$

8. Simili modo si statuamus

$$s = \int \frac{dx}{\sqrt{(1 - x^4)}} = v\sqrt{(1 - x^4)},$$

pervenietur ad hanc aequationem

$$\frac{dv}{dx}(1 - x^4) - 2vx^3 - 1 = 0,$$

ubi iam statuamus

$$v = \alpha x + \beta x^5 + \gamma x^9 + \delta x^{13} + \varepsilon x^{17} + \zeta x^{21} + \text{etc.},$$

cuius evolutio ita repraesentetur

$$\begin{aligned}
\frac{dv}{dx} &= \alpha + 5\beta x^4 + 9\gamma x^8 + 13\delta x^{12} + 17\varepsilon x^{16} + \text{etc.}\\
-\frac{x^4 dv}{dx} &= \quad -\alpha \quad -5\beta \quad -9\gamma \quad -13\delta \quad -\text{etc.}\\
-2vx^3 &= \quad -2\alpha \quad -2\beta \quad -2\gamma \quad -2\delta \quad -\text{etc.}\\
-1 &= -1
\end{aligned}$$

Hinc reperiuntur coefficientes

$$\alpha=1,\quad \beta=\frac{3}{5},\quad \gamma=\frac{3\cdot 7}{5\cdot 9},\quad \delta=\frac{3\cdot 7\cdot 11}{5\cdot 9\cdot 13},\quad \varepsilon=\frac{3\cdot 7\cdot 11\cdot 15}{5\cdot 9\cdot 13\cdot 17}\quad \text{etc.},$$

unde colligitur fore

$$s=\left(x+\frac{3}{5}x^5+\frac{3\cdot 7}{5\cdot 9}x^9+\frac{3\cdot 7\cdot 11}{5\cdot 9\cdot 13}x^{13}+\text{etc.}\right)\sqrt{(1-x^4)}.$$

His autem seriebus plane ad valores litterarum a et c eruendos uti non licet; facto enim $x=1$ formula $\sqrt{(1-x^4)}$ evanescit, tum autem ipsae series in infinitum excrescunt.

9. Pro litteris autem a et c cognoscendis alias adhiberi conveniet methodos inde petendas, quod integralia harum formularum

$$\int\frac{xxdx}{\sqrt{(1-x^4)}} \quad \text{et} \quad \int\frac{dx}{\sqrt{(1-x^4)}}$$

pro eo tantum casu quaeruntur, quo post integrationem fit $x=1$. Hunc in finem formula $\frac{1}{\sqrt{(1-x^4)}}$ ita repraesentetur

$$\frac{(1+xx)^{-\frac{1}{2}}}{\sqrt{(1-xx)}}$$

et numerator $(1+xx)^{-\frac{1}{2}}$ in seriem convertatur, quae erit

$$1-\frac{1}{2}xx+\frac{1}{2}\cdot\frac{3}{4}x^4-\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}x^6+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}x^8-\text{etc.},$$

ita ut loco $\frac{1}{\sqrt{(1-x^4)}}$ scripturi simus hanc seriem

$$\frac{1}{\sqrt{(1-xx)}}\left(1-\frac{1}{2}xx+\frac{1}{2}\cdot\frac{3}{4}x^4-\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}x^6+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}x^8-\text{etc.}\right),$$

quo facto tam pro y quam pro s sequentes formulae integrandae occurrent

$$\int\frac{dx}{\sqrt{(1-xx)}},\quad \int\frac{xxdx}{\sqrt{(1-xx)}},\quad \int\frac{x^4dx}{\sqrt{(1-xx)}}\quad \text{etc.}$$

10. Harum autem formularum integralia hic non in genere requiruntur, sed tantum pro casu, quo post integrationem ponitur $x = 1$. Hoc autem casu novimus, si $1:\pi$ denotet rationem diametri ad peripheriam, esse

$$\int\frac{dx}{\sqrt{(1-xx)}}=\frac{\pi}{2},\qquad \int\frac{xx\,dx}{\sqrt{(1-xx)}}=\frac{1}{2}\cdot\frac{\pi}{2},$$
$$\int\frac{x^4dx}{\sqrt{(1-xx)}}=\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{\pi}{2},\quad \int\frac{x^6dx}{\sqrt{(1-xx)}}=\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{\pi}{2}$$

et ita porro; quibus valoribus substitutis primo ex formula

$$y=\int\frac{xx\,dx}{\sqrt{(1-x^4)}}=\int\frac{xx\,dx(1+xx)^{-\frac{1}{2}}}{\sqrt{(1-xx)}}$$

colligimus fore

$$a=\frac{\pi}{2}\left(\frac{1}{2}-\frac{1^2}{2^2}\cdot\frac{3}{4}+\frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5}{6}-\frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}\cdot\frac{7}{8}+\text{etc.}\right),$$

ex altera autem formula

$$s=\int\frac{dx}{\sqrt{(1-x^4)}}$$

colligitur longitudo totius arcus

$$CA=c=\frac{\pi}{2}\left(1-\frac{1^2}{2^2}+\frac{1^2}{2^2}\cdot\frac{3^2}{4^2}-\frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}+\frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}\cdot\frac{7^2}{8^2}-\text{etc.}\right).$$

Verum etiam hae series non satis sunt aptae pro veris valoribus quantitatum a et c cognoscendis.

11. Superest autem adhuc alia methodus eosdem valores per producta ex infinitis factoribus exprimendi, cuius rationem, quamquam a me iam dudum [1]) fusius est explicata, hic sequenti modo succincte exponam. Consideretur haec formula $z = x^n\sqrt{(1-x^4)}$, et cum sit

$$dz=nx^{n-1}dx\sqrt{(1-x^4)}-\frac{2x^{n+3}dx}{\sqrt{(1-x^4)}}=\frac{nx^{n-1}dx-(n+2)x^{n+3}dx}{\sqrt{(1-x^4)}},$$

1) L. EULERI Commentatio 154 (indicis ENESTROEMIANI): *Animadversiones in rectificationem ellipsis*, Opusc. var. arg. 2, 1750, p. 121; *LEONHARDI EULERI Opera omnia*, series I, vol. 20; vide praecipue p. 25. A. K.

hinc vicissim integrando erit

$$x^n\sqrt{(1-x^4)}=n\int\frac{x^{n-1}dx}{\sqrt{(1-x^4)}}-(n+2)\int\frac{x^{n+3}dx}{\sqrt{(1-x^4)}};$$

quare si haec integralia tantum desiderentur pro casu $x=1$, fiet

$$\int\frac{x^{n-1}dx}{\sqrt{(1-x^4)}}=\frac{n+2}{n}\int\frac{x^{n+3}dx}{\sqrt{(1-x^4)}}.$$

Simili modo erit

$$\int\frac{x^{n+3}dx}{\sqrt{(1-x^4)}}=\frac{n+6}{n+4}\int\frac{x^{n+7}dx}{\sqrt{(1-x^4)}}$$

et

$$\int\frac{x^{n+7}dx}{\sqrt{(1-x^4)}}=\frac{n+10}{n+8}\int\frac{x^{n+11}dx}{\sqrt{(1-x^4)}}\quad\text{etc.}$$

Quodsi ergo hoc modo in infinitum ascendamus, erit

$$\int\frac{x^{n-1}dx}{\sqrt{(1-x^4)}}=\frac{n+2}{n}\cdot\frac{n+6}{n+4}\cdot\frac{n+10}{n+8}\cdot\frac{n+14}{n+12}\cdots\int\frac{x^{n+\infty}dx}{\sqrt{(1-x^4)}}.$$

12. Substituamus nunc succesive pro n numeros 1, 2, 3, 4 ac prodibunt sequentes quatuor reductiones ad producta infinita, casu scilicet $x=1$.

$$\text{I.}\quad\int\frac{dx}{\sqrt{(1-x^4)}}=\frac{3}{1}\cdot\frac{7}{5}\cdot\frac{11}{9}\cdot\frac{15}{13}\cdot\frac{19}{17}\cdots\int\frac{x^{1+\infty}dx}{\sqrt{(1-x^4)}}=c$$

$$\text{II.}\quad\int\frac{xdx}{\sqrt{(1-x^4)}}=\frac{4}{2}\cdot\frac{8}{6}\cdot\frac{12}{10}\cdot\frac{16}{14}\cdot\frac{20}{18}\cdots\int\frac{x^{2+\infty}dx}{\sqrt{(1-x^4)}}=\frac{\pi}{4}$$

$$\text{III.}\quad\int\frac{xxdx}{\sqrt{(1-x^4)}}=\frac{5}{3}\cdot\frac{9}{7}\cdot\frac{13}{11}\cdot\frac{17}{15}\cdot\frac{21}{19}\cdots\int\frac{x^{3+\infty}dx}{\sqrt{(1-x^4)}}=a$$

$$\text{IV.}\quad\int\frac{x^3dx}{\sqrt{(1-x^4)}}=\frac{6}{4}\cdot\frac{10}{8}\cdot\frac{14}{12}\cdot\frac{18}{16}\cdot\frac{22}{20}\cdots\int\frac{x^{4+\infty}dx}{\sqrt{(1-x^4)}}=\frac{1}{2}.$$

13. Hic iam probe notandum est postremas formulas integrales inter se omnes esse aequales. Cum enim in genere sit

$$\int\frac{x^{n-1}dx}{\sqrt{(1-x^4)}}=\frac{n+2}{n}\int\frac{x^{n+3}dx}{\sqrt{(1-x^4)}},$$

sumto $n = \infty$ erit
$$\int \frac{x^{\infty-1}dx}{\sqrt{(1-x^4)}} = \int \frac{x^{\infty+3}dx}{\sqrt{(1-x^4)}}.$$

Quodsi ergo harum quatuor formularum quamlibet per aliam dividamus, postremi factores integrales se mutuo tollunt eritque

$$\frac{\mathrm{I}}{\mathrm{II}} = \frac{4c}{\pi} = \frac{2\cdot 3}{1\cdot 4}\cdot\frac{6\cdot 7}{5\cdot 8}\cdot\frac{10\cdot 11}{9\cdot 12}\cdot\frac{14\cdot 15}{13\cdot 16}\cdot\frac{18\cdot 19}{17\cdot 20}\ \text{etc.}$$
$$\frac{\mathrm{I}}{\mathrm{III}} = \frac{c}{a} = \frac{3\cdot 3}{1\cdot 5}\cdot\frac{7\cdot 7}{5\cdot 9}\cdot\frac{11\cdot 11}{9\cdot 13}\cdot\frac{15\cdot 15}{13\cdot 17}\cdot\frac{19\cdot 19}{17\cdot 21}\ \text{etc.}$$
$$\frac{\mathrm{I}}{\mathrm{IV}} = 2c = \frac{3\cdot 4}{1\cdot 6}\cdot\frac{7\cdot 8}{5\cdot 10}\cdot\frac{11\cdot 12}{9\cdot 14}\cdot\frac{15\cdot 16}{13\cdot 18}\cdot\frac{19\cdot 20}{17\cdot 22}\ \text{etc.}$$
$$\frac{\mathrm{II}}{\mathrm{III}} = \frac{\pi}{4a} = \frac{3\cdot 4}{2\cdot 5}\cdot\frac{7\cdot 8}{6\cdot 9}\cdot\frac{11\cdot 12}{10\cdot 13}\cdot\frac{15\cdot 16}{14\cdot 17}\cdot\frac{19\cdot 20}{18\cdot 21}\ \text{etc.}$$
$$\frac{\mathrm{II}}{\mathrm{IV}} = \frac{\pi}{2} = \frac{4\cdot 4}{2\cdot 6}\cdot\frac{8\cdot 8}{6\cdot 10}\cdot\frac{12\cdot 12}{10\cdot 14}\cdot\frac{16\cdot 16}{14\cdot 18}\cdot\frac{20\cdot 20}{18\cdot 22}\ \text{etc.}$$
$$\frac{\mathrm{III}}{\mathrm{IV}} = 2a = \frac{4\cdot 5}{3\cdot 6}\cdot\frac{8\cdot 9}{7\cdot 10}\cdot\frac{12\cdot 13}{11\cdot 14}\cdot\frac{16\cdot 17}{15\cdot 18}\cdot\frac{20\cdot 21}{19\cdot 22}\ \text{etc.}$$

14. Hae iam expressiones multo sunt aptiores ad veros valores litterarum a et c proxime definiendos. Pro valore autem ipsius c inveniendo formula $\frac{\mathrm{I}}{\mathrm{II}}$ maxime videtur idonea, unde fit

$$\frac{4c}{\pi} = \frac{1\cdot 3}{2\cdot 1}\cdot\frac{3\cdot 7}{4\cdot 5}\cdot\frac{5\cdot 11}{6\cdot 9}\cdot\frac{7\cdot 15}{8\cdot 13}\cdot\frac{9\cdot 19}{10\cdot 17}\ \text{etc.}$$

sive

$$\frac{4c}{\pi} = \frac{3}{2}\cdot\frac{21}{20}\cdot\frac{55}{54}\cdot\frac{105}{104}\cdot\frac{171}{170}\ \text{etc.},$$

quae pro faciliori calculo ita potest exhiberi

$$\frac{4c}{\pi} = \left(1+\frac{1}{2}\right)\left(1+\frac{1}{20}\right)\left(1+\frac{1}{54}\right)\left(1+\frac{1}{104}\right)\left(1+\frac{1}{170}\right)\ \text{etc.}$$

At vero quantitas a commodissime definietur sive ex hac forma

$$\frac{\pi}{4a} = \frac{2\cdot 3}{1\cdot 5}\cdot\frac{4\cdot 7}{3\cdot 9}\cdot\frac{6\cdot 11}{5\cdot 13}\cdot\frac{8\cdot 15}{7\cdot 17}\cdot\frac{10\cdot 19}{9\cdot 21}\ \text{etc.}$$

sive

$$\frac{\pi}{4a} = \frac{6}{5}\cdot\frac{28}{27}\cdot\frac{66}{65}\cdot\frac{120}{119}\cdot\frac{190}{189}\ \text{etc.},$$

quae commode ergo ita repraesentetur

$$\frac{\pi}{4a}=\left(1+\frac{1}{1\cdot 5}\right)\left(1+\frac{1}{3\cdot 9}\right)\left(1+\frac{1}{5\cdot 13}\right)\left(1+\frac{1}{7\cdot 17}\right)\left(1+\frac{1}{9\cdot 21}\right)\text{ etc.};$$

vel etiam pari successu definietur quantitas a ex formula $\frac{\text{III}}{\text{IV}}$, quae dat

$$2a=\frac{2\cdot 5}{3\cdot 3}\cdot\frac{4\cdot 9}{5\cdot 7}\cdot\frac{6\cdot 13}{7\cdot 11}\cdot\frac{8\cdot 17}{9\cdot 15}\cdot\frac{10\cdot 21}{11\cdot 19}\text{ etc.}$$

sive

$$2a=\frac{10}{9}\cdot\frac{36}{35}\cdot\frac{78}{77}\cdot\frac{136}{135}\cdot\frac{210}{209}\text{ etc.}$$

sive

$$2a=\left(1+\frac{1}{3\cdot 3}\right)\left(1+\frac{1}{5\cdot 7}\right)\left(1+\frac{1}{7\cdot 11}\right)\left(1+\frac{1}{9\cdot 15}\right)\left(1+\frac{1}{11\cdot 19}\right)\text{ etc.}$$

Interim tamen satis taedioso calculo opus foret, si valores harum litterarum usque ad partem millionesimam unitatis iustos exquirere vellemus; verum infra, cum proprietates magis absconditas huius curvae detexerimus, satis prompte hos valores exhibere licebit.

15. At vero pro eodem scopo series pro a et c supra § 10 inventae optimo cum successu usurpari possunt, quanquam ipsi termini parum decrescunt, propterea quod in istis seriebus signa $+$ et $-$ alternantur. Hinc enim insigne subsidium nascitur ad summas harum serierum proxime inveniendas. Si enim habeatur huiusmodi series

$$A-A'+A''-A'''+A''''-A'''''\text{ etc.},$$

cuius termini A, A', A'', A''' continuo fiant minores, tum inde formetur series differentiarum

$$A-A'=B,\quad A'-A''=B',\quad A''-A'''=B''\text{ etc.}$$

hincque porro series differentiarum secundarum

$$B-B'=C,\quad B'-B''=C',\quad B''-B'''=C''\text{ etc.}$$

sicque hoc modo continuo differentiae capiantur, tum summa seriei propositae semper erit

$$\frac{A}{2}+\frac{B}{4}+\frac{C}{8}+\frac{D}{16}+\frac{E}{32}+\text{ etc.}$$

16. Quo nunc hanc regulam ad series § 10 applicemus, evolvamus in fractionibus decimalibus singulos terminos, qui ibi occurrunt.

$$\frac{1}{2} = 0{,}500000$$

$$\frac{1^2}{2^2} = 0{,}250000$$

$$\frac{1^2}{2^2} \cdot \frac{3}{4} = 0{,}187500$$

$$\frac{1^2}{2^2} \cdot \frac{3^2}{4^2} = 0{,}140625$$

$$\frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5}{6} = 0{,}117188$$

$$\frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} = 0{,}097657$$

$$\frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdot \frac{7}{8} = 0{,}085450$$

$$\frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdot \frac{7^2}{8^2} = 0{,}074769$$

$$\frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdot \frac{7^2}{8^2} \cdot \frac{9}{10} = 0{,}067292$$

$$\frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdot \frac{7^2}{8^2} \cdot \frac{9^2}{10^2} = 0{,}060563$$

$$\frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdot \frac{7^2}{8^2} \cdot \frac{9^2}{10^2} \cdot \frac{11}{12} = 0{,}055516$$

$$\frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdot \frac{7^2}{8^2} \cdot \frac{9^2}{10^2} \cdot \frac{11^2}{12^2} = 0{,}050890$$

$$\frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdot \frac{7^2}{8^2} \cdot \frac{9^2}{10^2} \cdot \frac{11^2}{12^2} \cdot \frac{13}{14} = 0{,}047255$$

$$\frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdot \frac{7^2}{8^2} \cdot \frac{9^2}{10^2} \cdot \frac{11^2}{12^2} \cdot \frac{13^2}{14^2} = 0{,}043880$$

$$\frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdot \frac{7^2}{8^2} \cdot \frac{9^2}{10^2} \cdot \frac{11^2}{12^2} \cdot \frac{13^2}{14^2} \cdot \frac{15}{16} = 0{,}041138$$

$$\frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdot \frac{7^2}{8^2} \cdot \frac{9^2}{10^2} \cdot \frac{11^2}{12^2} \cdot \frac{13^2}{14^2} \cdot \frac{15^2}{16^2} = 0{,}038567$$

$$\frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdot \frac{7^2}{8^2} \cdot \frac{9^2}{10^2} \cdot \frac{11^2}{12^2} \cdot \frac{13^2}{14^2} \cdot \frac{15^2}{16^2} \cdot \frac{17}{18} = 0{,}036424$$

$$\frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}\cdot\frac{7^2}{8^2}\cdot\frac{9^2}{10^2}\cdot\frac{11^2}{12^2}\cdot\frac{13^2}{14^2}\cdot\frac{15^2}{16^2}\cdot\frac{17^2}{18^2}=0{,}034400\,^{1)}$$

$$\frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}\cdot\frac{7^2}{8^2}\cdot\frac{9^2}{10^2}\cdot\frac{11^2}{12^2}\cdot\frac{13^2}{14^2}\cdot\frac{15^2}{16^2}\cdot\frac{17^2}{18^2}\cdot\frac{19}{20}=0{,}032700\,^{1)}$$

$$\frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}\cdot\frac{7^2}{8^2}\cdot\frac{9^2}{10^2}\cdot\frac{11^2}{12^2}\cdot\frac{13^2}{14^2}\cdot\frac{15^2}{16^2}\cdot\frac{17^2}{18^2}\cdot\frac{19^2}{20^2}=0{,}031065\,^{1)}.$$

17. His praeparatis calculum instituamus pro valore litterae c inveniendo, et cum esset

$$\frac{2c}{\pi}=1-\frac{1^2}{2^2}+\frac{1^2}{2^2}\cdot\frac{3^2}{4^2}-\frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}+\text{etc.},$$

binis primis terminis ad sinistram translatis erit

$$\frac{2c}{\pi}-\frac{3}{4}=\frac{1^2}{2^2}\cdot\frac{3^2}{4^2}-\frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}+\frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}\cdot\frac{7^2}{8^2}\ \text{etc.}$$

Nunc singuli huius seriei termini sibi invicem subscribantur iisque subiungantur series differentiarum litteris B, C, D etc. insignitarum hoc modo:

A	B	C	D	E	F	G	H
0,140625	0,042968	0,020080	0,011398	0,007249	0,004970	0,003595	0,002709
0,097657	0,022888	0,008682	0,004149	0,002279	0,001375	0,000886	0,000575
0,074769	0,014206	0,004533	0,001870	0,000904	0,000489	0,000311	
0,060563	0,009673	0,002663	0,000966	0,000415	0,000178		
0,050890	0,007010	0,001697	0,000551	0,000237			
0,043880	0,005313	0,001146	0,000314				
0,038567	0,004167	0,000832					
0,034400	0,003335						
0,031065							

1) Tres ultimi numeri revera valores 0,034399; 0,032679; 0,031045 habent; errores corrigere neglleximus, cum nullius sint momenti. A. K.

18. Hinc igitur summa nostrae seriei sequenti modo colligetur:

$\frac{1}{2}A = 0{,}070312$	$0{,}084503$
$\frac{1}{4}B = 0{,}010742$	$\frac{1}{64}F = 0{,}000078$
$\frac{1}{8}C = 0{,}002510$	$\frac{1}{128}G = 0{,}000028$
$\frac{1}{16}D = 0{,}000712$	$\frac{1}{256}H = 0{,}000011$
$\frac{1}{32}E = 0{,}000227$	pro reliquis $0{,}000007$
$0{,}084503$	$0{,}084627$
	adde $\frac{3}{4} = 0{,}750000$
	erit $\frac{2c}{\pi} = 0{,}834627$.

Hinc ergo erit

$$c = \pi \cdot 0{,}417314 = 1{,}311031.$$

19. Simili modo computabitur intervallum $AB = CD = a$. Erat autem

$$\frac{2a}{\pi} = \frac{1}{2} - \frac{1^2}{2^2}\cdot\frac{3}{4} + \frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5}{6} - \frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}\cdot\frac{7}{8} + \text{etc.},$$

ubi bini primi termini

$$\frac{1}{2} - \frac{3}{16} = \frac{5}{16} = 0{,}312500$$

dant ad alteram partem translati

$$\frac{2a}{\pi} - 0{,}312500 = \frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5}{6} - \frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}\cdot\frac{7}{8} + \text{etc.},$$

unde calculus sequenti modo expediatur:

A	B	C	D	E	F	G	H
0,117188	0,031738	0,013580	0,007198	0,004331	0,002835	0,001969	0,001405
0,085450	0,018158	0,006382	0,002867	0,001496	0,000866	0,000564	
0,067292	0,011776	0,003515	0,001371	0,000630	0,000302		
0,055516	0,008261	0,002144	0,000741	0,000328			
0,047255	0,006117	0,001403	0,000413				
0,041138	0,004714	0,000990					
0,036424	0,003724						
0,032700							

20. Hinc igitur seriei summa colligitur

$$\begin{aligned}
\frac{1}{2}A &= 0,058594 \\
\frac{1}{4}B &= 0,007934 \\
\frac{1}{8}C &= 0,001697 \\
\frac{1}{16}D &= 0,000450 \\
\frac{1}{32}E &= 0,000135 \\
&\overline{0,068810}
\end{aligned}
\qquad\qquad
\begin{aligned}
& 0,068810 \\
\frac{1}{64}F &= 0,000044 \\
\frac{1}{128}G &= 0,000015 \\
\frac{1}{256}H &= 0,000005 \\
&\overline{0,068874} \\
\text{adde } \frac{5}{16} &= 0,312500 \\
\text{et prodit } \frac{2a}{\pi} &= 0,381374,
\end{aligned}$$

hinc ergo

$$a = \pi \cdot 0,190687 = 0,599061.$$

20[a] [1]). His valoribus quantitatum a et c proxime veris inventis, quos autem deinceps adhuc accuratius definire docebo, progredior ad illas proprietates huius curvae magis abstrusas, quas sum pollicitus demonstrandas, quippe quas per solitas calculi operationes vix ac ne vix quidem eruere licet et quae propterea profundioris indaginis merito sunt censendae. Ac primo qui-

1) In editione principe falso numerus 20 iteratur. A. K.

dem hic eam insignem relationem, quae inter ternas principales dimensiones huius curvae, scilicet altitudinem $BC = AD$ et inter latitudinem $AB = CD$ atque ipsam curvae longitudinem AMC intercedit et quam iam pridem detexi, hic accuratius exponam et sequenti theoremate complectar.

THEOREMA 1

21. *In curva elastica rectangula AMC, cuius vertex est A et centrum alternationis C, ternae dimensiones principales, quae sunt 1) altitudo $BC = AD$, 2) latitudo $AB = CD$ ac 3) longitudo arcus AMC, ita a se invicem pendent, ut rectangulum ex latitudine AB in longitudinem arcus AMC aequale sit areae circuli circa diametrum altitudinis BC descripti sive positis, ut fecimus, $BC = AD = 1$, $AB = CD = a$ et arcu $AMC = c$ erit $ac = \frac{\pi}{4}$.*

DEMONSTRATIO

22. Insignis ista proprietas deducitur ex formulis, quas supra per producta in infinitum excurrentia expressimus (§ 13), quarum prima dabat

$$\frac{4c}{\pi} = \frac{2\cdot 3}{1\cdot 4}\cdot\frac{6\cdot 7}{5\cdot 8}\cdot\frac{10\cdot 11}{9\cdot 12}\cdot\frac{14\cdot 15}{13\cdot 16} \text{ etc.},$$

ultima vero

$$2a = \frac{4\cdot 5}{3\cdot 6}\cdot\frac{8\cdot 9}{7\cdot 10}\cdot\frac{12\cdot 13}{11\cdot 14}\cdot\frac{16\cdot 17}{15\cdot 18} \text{ etc.}$$

Quodsi iam in priore expressione primus factor simplex $\frac{2}{1}$ seorsim exhibeatur, ex reliquis autem sequentibus bini inter se combinentur, habebitur

$$\frac{4c}{\pi} = \frac{2}{1}\cdot\frac{3\cdot 6}{4\cdot 5}\cdot\frac{7\cdot 10}{8\cdot 9}\cdot\frac{11\cdot 14}{12\cdot 13}\cdot\frac{15\cdot 18}{16\cdot 17} \text{ etc.}$$

Quodsi ergo haec expressio per alteram multiplicetur, omnes factores praeter primum manifesto se mutuo tollunt, ita ut proditurum sit $\frac{8ac}{\pi} = 2$, unde fit

$$ac = \frac{\pi}{4},$$

quae est ipsa illa proprietas, quam demonstrare oportebat.

23. Etsi haec veritas modo prorsus singulari ex contemplatione infiniti est conclusa, tamen deinceps observavi eandem quoque per operationes calculi magis consuetas elici posse. Quaeramus enim in genere pro quovis curvae puncto indefinito M productum ex applicata $PM=y$ et arcu $CM=s$ sitque hoc productum $P=ys$; erit $dP=yds+sdy$ hincque iterum integrando

$$P=\int yds+\int sdy,$$

quas ambas formulas seorsim evolvamus. Pro priori initio ostendimus esse

$$y=\frac{1}{3}x^3+\frac{1\cdot1}{2\cdot7}x^7+\frac{1\cdot3\cdot1}{2\cdot4\cdot11}x^{11}+\frac{1\cdot3\cdot5\cdot1}{2\cdot4\cdot6\cdot15}x^{15}+\text{etc.},$$

quae series ducta in $ds=\frac{dx}{\sqrt{(1-x^4)}}$ per singulos terminos ita integretur, ut post integrationem statuatur $x=1$, quippe in quo versatur casus nostri theorematis.

24. Pro hac autem investigatione habebimus

$$\int\frac{x^3dx}{\sqrt{(1-x^4)}}=\frac{1}{2}-\frac{1}{2}\sqrt{(1-x^4)}=\frac{1}{2}$$

posito $x=1$; tum vero in genere vidimus esse (§ 11)

$$\int\frac{x^{n+3}dx}{\sqrt{(1-x^4)}}=\frac{n}{n+2}\int\frac{x^{n-1}dx}{\sqrt{(1-x^4)}},$$

unde deducimus

$$\int\frac{x^7dx}{\sqrt{(1-x^4)}}=\frac{2}{3}\cdot\frac{1}{2}$$

$$\int\frac{x^{11}dx}{\sqrt{(1-x^4)}}=\frac{8}{10}\cdot\frac{4}{6}\cdot\frac{1}{2}=\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{1}{2}$$

$$\int\frac{x^{15}dx}{\sqrt{(1-x^4)}}=\frac{12}{14}\cdot\frac{8}{10}\cdot\frac{4}{6}\cdot\frac{1}{2}=\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{1}{2}.$$

Hinc igitur pro nostro casu, quo $x=1$, erit

$$\int yds=\frac{1}{3}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{7}\cdot\frac{2}{3}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{11}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{1}{2}$$
$$+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{15}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{1}{19}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}\cdot\frac{1}{2}+\text{etc.},$$

quae series reducitur ad sequentem formam

$$\int y\,ds = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{3\cdot 7} + \frac{1}{5\cdot 11} + \frac{1}{7\cdot 15} + \frac{1}{9\cdot 19} + \text{etc.}\right).$$

Eodem modo evolvatur altera formula $\int s\,dy$, et cum per seriem priorem esset

$$s = x + \frac{1}{2}\cdot\frac{1}{5}x^5 + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{9}x^9 + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{13}x^{13} + \text{etc.},$$

at vero $dy = \frac{xx\,dx}{\sqrt{(1-x^4)}}$, singulis terminis integrandis ope formularum ante datarum pro casu $x = 1$ reperietur

$$\int s\,dy = \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{5}\cdot\frac{2}{3}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{9}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{13}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{1}{2},$$

quae series contrahitur in sequentem formam

$$\int s\,dy = \frac{1}{2}\left(1 + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 9} + \frac{1}{7\cdot 13} + \frac{1}{9\cdot 17} + \frac{1}{11\cdot 21} + \text{etc.}\right).$$

His igitur duabus seriebus coniunctis fiet

$$P = \frac{1}{2}\left(1 + \frac{1}{3} + \frac{1}{3\cdot 5} + \frac{1}{3\cdot 7} + \frac{1}{5\cdot 9} + \frac{1}{5\cdot 11} + \frac{1}{7\cdot 13} + \text{etc.}\right).$$

25. Quodsi in hac serie bini termini se insequentes in unum contrahantur, obtinebitur sequens series

$$P = ys = \frac{2}{1\cdot 3} + \frac{2}{5\cdot 7} + \frac{2}{9\cdot 11} + \frac{2}{13\cdot 15} + \frac{2}{17\cdot 19} + \text{etc.}$$

Quoniam autem porro est

$$\frac{2}{3} = 1 - \frac{1}{3} \quad \text{et} \quad \frac{2}{5\cdot 7} = \frac{1}{5} - \frac{1}{7}, \quad \frac{2}{9\cdot 11} = \frac{1}{9} - \frac{1}{11} \quad \text{etc.},$$

ista series resolvitur in hanc formam

$$P = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \text{etc.};$$

quae cum sit notissima series LEIBNIZIANA, cuius summa $=\frac{\pi}{4}$, erit

$$P = ys = \frac{\pi}{4},$$

casu scilicet, quo $x=1$. Verum hoc casu assumsi fieri $y=a$ et $s=c$ sicque etiam hinc apparet esse productum $ac=\frac{\pi}{4}$.

PRAEPARATIO AD SEQUENTES HUIUS CURVAE PROPRIETATES MAGIS ABSTRUSAS

26. In dissertatione, cui titulus *Plenior explicatio circa comparationem quantitatum in formula integrali*

$$\int\frac{Zdz}{\sqrt{(1+mzz+nz^4)}}$$

contentarum quaeque parti posteriori Actorum pro anno 1781[1]) inserta fuit, ostendi, si $\Pi:z$ denotet valorem huius formulae integralis

$$\int\frac{dz(\alpha+\beta zz)}{\sqrt{(1+mzz+nz^4)}}$$

ita sumtum, ut evanescat posito $z=0$, tum plures huius generis quantitates transcendentes modo prorsus singulari inter se comparari posse. Scilicet si propositae fuerint duae huiusmodi formulae $\Pi:x$ et $\Pi:y$ atque ex litteris x et y ita determinetur tertia z, ut sit

$$z=\frac{x\sqrt{(1+myy+ny^4)}}{1-nxxyy}+\frac{y\sqrt{(1+mxx+nx^4)}}{1-nxxyy},$$

unde fit

$$\sqrt{(1+mzz+nz^4)}$$
$$=\frac{(mxy+\sqrt{(1+mxx+nx^4)(1+myy+ny^4)})(1+nxxyy)+2nxy(xx+yy)}{(1-nxxyy)^2},$$

tum semper erit

$$\Pi:z=\Pi:x+\Pi:y+\beta xyz,$$

ita ut quantitas transcendens $\Pi:z$ superet summam datarum $\Pi:x$ et $\Pi:y$ quantitate algebraica βxyz.

1) L. EULERI Commentatio 581 (indicis ENESTROEMIANI); vide p. 39. A. K.

27. Evidens iam est has formulas generales duplici modo ad institutum nostrum accommodari posse, scilicet tam ad arcus huius curvae inter se comparandos quam ad applicatas cuique abscissae z respondentes. Pro utroque casu autem erit $m=0$ et $n=-1$, tum vero in numeratore pro arcubus sumi debebit $\alpha=1$ et $\beta=0$, at pro applicatis $\alpha=0$ et $\beta=1$.

28. Quodsi iam littera z denotet abscissam quamcunque in axe CB assumtam, applicatam ei respondentem designemus charactere $\Pi:z$, arcum vero respondentem hoc charactere $\Theta:z$ eritque ex natura nostrae elasticae

$$\Pi:z=\int\frac{zzdz}{\sqrt{(1-z^4)}}\quad\text{et}\quad\Theta:z=\int\frac{dz}{\sqrt{(1-z^4)}},$$

quibus characteribus in sequentibus utemur. Tum igitur sumto $z=0$ erit

$$\Pi:0=0\quad\text{et}\quad\Theta:0=0.$$

Sumto autem $z=1$ erit

$$\Pi:1=AB=a\quad\text{et}\quad\Theta:1=CA=c.$$

Praeterea vero notari oportet sumta abscissa z negativa tam applicatam quam arcus longitudinem etiam fore negativas; sicque erit $\Pi:(-z)=-\Pi:z$ similique modo $\Theta:(-z)=-\Theta:z$. His igitur praemissis duplicem istam comparationem in sequentibus problematibus ad nostrum institutum accommodabimus.

PROBLEMA 1

29. *Propositis in nostra curva elastica binis arcubus CX et CY* (Fig. 4) *abscindere arcum CZ, qui aequalis sit summae arcuum $CX+CY$.*

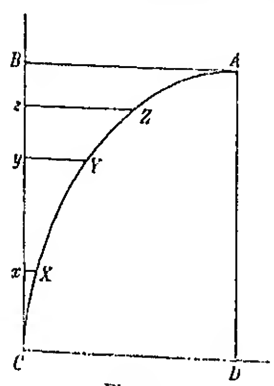

Fig. 4.

SOLUTIO

Vocentur abscissae his arcubus respondentes $Cx=x$, $Cy=y$ et $Cz=z$ eruntque applicatae stabilito signandi modo $xX=\Pi:x$, $yY=\Pi:y$ et $zZ=\Pi:z$, ipsi vero arcus $CX=\Theta:x$, $CY=\Theta:y$ et $CZ=\Theta:z$, et quoniam requiritur, ut sit $\Theta:z=\Theta:x+\Theta:y$, regula generalis supra allata, quoniam hoc casu littera $\beta=0$, pro datis litteris x et y ita definire iubet z, ut sit

$$z=\frac{x\sqrt{(1-y^4)}+y\sqrt{(1-x^4)}}{1+xxyy};$$

tum autem erit

$$\sqrt{(1-z^4)}=\frac{(1-xxyy)\sqrt{(1-x^4)(1-y^4)}-2xy(xx+yy)}{(1+xxyy)^2},$$

unde patet, quomodo ex binis abscissis datis $Cx=x$ et $Cy=y$ quaesitam z construi oporteat, ut arcus CZ aequalis fiat summae arcuum $CX+CY$.

30. Quemadmodum hic ex datis abscissis x et y determinavimus abscissam z, ita vicissim, si dentur abscissae x et z, tertia y ex iis simili modo determinabitur. Cum enim hic esse debeat $\Theta:y=\Theta:z-\Theta:x$, evidens est hic y eodem modo per z et $-x$ definiri, quo ante z per $+x$ et $+y$ expressimus. Hinc igitur erit

$$y=\frac{z\sqrt{(1-x^4)}-x\sqrt{(1-z^4)}}{1+xxzz}$$

et

$$\sqrt{(1-y^4)}=\frac{(1-xxzz)\sqrt{(1-x^4)(1-z^4)}+2xz(xx+zz)}{(1+xxzz)^2}.$$

Parique modo ex datis y et z abscissa x ita determinabitur, ut sit

$$x=\frac{z\sqrt{(1-y^4)}-y\sqrt{(1-z^4)}}{1+yyzz}$$

et

$$\sqrt{(1-x^4)}=\frac{(1-yyzz)\sqrt{(1-y^4)(1-z^4)}+2yz(yy+zz)}{(1+yyzz)^2}.$$

31. Hinc igitur patet ternas quantitates x, y et z ita inter se referri, ut quaelibet per binas reliquas simili fere modo determinetur; quamobrem istam relationem accuratius evolvamus, quo clarius pateat, quomodo a se invicem pendeant. Ex primis autem valoribus sumtis quadratis erit

$$zz=\frac{(xx+yy)(1-xxyy)+2xy\sqrt{(1-x^4)(1-y^4)}}{(1+xxyy)^2};$$

ex valore autem formulae $\sqrt{(1-z^4)}$ colligitur

$$\sqrt{(1-x^4)(1-y^4)}=\frac{(1+xxyy)^2\sqrt{(1-z^4)}+2xy(xx+yy)}{1-xxyy};$$

qui valor si ibi substituatur, orietur haec aequatio

$$zz(1-xxyy)=xx+yy+2xy\sqrt{(1-z^4)}.$$

Similique modo ex binis reliquis determinationibus fiet

$$yy(1-xxzz)=zz+xx-2xz\sqrt{(1-y^4)}$$

et

$$xx(1-yyzz)=yy+zz-2yz\sqrt{(1-x^4)}.$$

32. Quodsi has aequationes ab omni irrationalitate liberemus, ex singulis eadem resultabit aequatio rationalis, quae erit

$$\left.\begin{array}{l} +x^4-2xxyy+2x^4yyzz+x^4y^4z^4 \\ +y^4-2xxzz+2xxy^4zz \\ +z^4-2yyzz+2xxyyz^4 \end{array}\right\}=0$$

quaeque etiam ita exhiberi potest

$$0=\left\{\begin{array}{l} x^4+y^4+z^4-2xxyy-2xxzz-2yyzz \\ \quad +2xxyyzz(xx+yy+zz)+x^4y^4z^4 \end{array}\right.$$

ubi iam manifesto ternae litterae x, y et z aequaliter ingrediuntur; quoniam enim hic litterarum x, y, z tantum quadrata insunt, perinde est, sive eae negative capiantur sive positive.

33. Quoties ergo ternae abscissae $Cx=x$, $Cy=y$ et $Cz=z$ eam inter se tenent rationem, quam assignavimus, tum arcus CZ semper aequabitur summae binorum reliquorum CX et CY. Cum igitur hinc sit $CZ-CY=CX$, erit arcus $YZ=CX$, unde, si puncta Y et Z pro lubitu accipiantur, a puncto C semper arcus CX abscindi poterit, qui arcui YZ erit aequalis. Ac vicissim proposito arcu CX a puncto quovis dato Y abscindi poterit arcus YZ illi arcui CX aequalis. Sin autem terminus Z ut datus spectetur, ab eo retro abscindi poterit arcus ZY ipsi CX aequalis; quae cum sint satis obvia, superfluum foret pro iis peculiaria problemata constituere.

THEOREMA 2

34. *Si ternae abscissae $Cx=x$, $Cy=y$, $Cz=z$ ita fuerint assumtae, ut arcus CZ aequetur summae CX et CY, tum ternae applicatae $xX=\Pi:x$, $yY=\Pi:y$, $zZ=\Pi:z$ ita inter se erunt relatae, ut sit*

$$\Pi:z=\Pi:x+\Pi:y+xyz,$$

sive erit

$$zZ=xX+yY+\frac{Cx\cdot Cy\cdot Cz}{CB^2}.$$

DEMONSTRATIO

35. Cum relatio inter formulas $\Pi:x$, $\Pi:y$ et $\Pi:z$ eandem relationem inter abscissas x, y et z praebeat, quam pro formulis $\Theta:x$, $\Theta:y$ et $\Theta:z$ assignavimus, quoniam pro hoc casu littera β in forma generali adhibita unitati aequatur, vi relationis generalis erit

$$\Pi:z=\Pi:x+\Pi:y+xyz,$$

unde ad homogeneitatem observandam, quia altitudo CB unitate est definita, solidum xyz per eius quadratum dividi oportet, unde fiet

$$zZ=xX+yY+\frac{Cx\cdot Cy\cdot Cz}{CB^2}.$$

36. Cum igitur characteres $\Theta:z$ et $\Pi:z$ certas functiones transcendentes abscissae z denotent, quas constat neque per logarithmos neque per arcus circulares exprimi posse, quandoquidem per formulas integrales $\int\frac{dz}{\sqrt{(1-z^4)}}$ et $\int\frac{zzdz}{\sqrt{(1-z^4)}}$ definiuntur, earum valores saltem per series infinitas exhibuisse iuvabit; erit autem per modum priorem

$$\Theta:z=z+\frac{1}{2}\cdot\frac{1}{5}z^5+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{9}z^9+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{13}z^{13}+\text{etc.}$$

et

$$\Pi:z=\frac{1}{3}z^3+\frac{1}{2}\cdot\frac{1}{7}z^7+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{11}z^{11}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{15}z^{15}+\text{etc.}$$

Ex altera autem resolutione erit ex § 8

$$\Theta : z = \left(z + \frac{3}{5}z^5 + \frac{3\cdot 7}{5\cdot 9}z^9 + \frac{3\cdot 7\cdot 11}{5\cdot 9\cdot 13}z^{13} + \text{etc.}\right)\sqrt{(1-z^4)}$$

et

$$\Pi : z = \left(\frac{1}{3}z^3 + \frac{1\cdot 5}{3\cdot 7}z^7 + \frac{1\cdot 5\cdot 9}{3\cdot 7\cdot 11}z^{11} + \frac{1\cdot 5\cdot 9\cdot 13}{3\cdot 7\cdot 11\cdot 15}z^{15} + \text{etc.}\right)\sqrt{(1-z^4)}.$$

PROBLEMA 2

37. *Elementa principalia nostrae curvae elasticae, scilicet latitudinem $AB = a$ et totum arcum $CA = c$, respectu altitudinis $CB = 1$ accuratius determinare, quam supra fieri licuit.*

SOLUTIO

Hunc in finem accipiatur punctum Z in ipso vertice curvae A, ut fiat $z = 1$, eritque

$$\Pi : z = AB = a \quad \text{et} \quad \Theta : z = CA = c;$$

tum igitur erit $\sqrt{(1-z^4)} = 0$. Nunc quaerantur bini arcus CX et CY, quorum summa sit aequalis arcui $CA = c$. Positis ergo eorum abscissis $Cx = x$ et $Cy = y$ ex § 31 erit

$$1 - xx - yy - xxyy = 0,$$

unde fit

$$yy = \frac{1-xx}{1+xx}.$$

Quodsi igitur y hoc modo per x determinetur, tum erit

$$\Theta : x + \Theta : y = c;$$

tum vero ob $\Pi : z = a$ erit

$$a = \Pi : x + \Pi : y + xy.$$

38. Quo nunc series pro $\Theta : x$ et $\Theta : y$, item pro $\Pi : x$ et $\Pi : y$ maxime convergentes reddantur, abscissas x et y proxime inter se aequales accipiamus. Si enim vellemus statuere $y = x$, prodiret $x = y = \sqrt{(-1 + \sqrt{2})}$, qui valor irrationalis minime idoneus foret ad nostras series evolvendas.

Hanc ob rem sumamus $xx = \frac{1}{2}$; erit $yy = \frac{1}{3}$ ideoque $x = \frac{1}{\sqrt{2}}$ et $y = \frac{1}{\sqrt{3}}$, unde per priores series fiet

$$\Theta : x = \frac{1}{\sqrt{2}}\left(1 + \frac{1}{2\cdot 5}\cdot\frac{1}{2^2} + \frac{1\cdot 3}{2\cdot 4\cdot 9}\cdot\frac{1}{2^4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 13}\cdot\frac{1}{2^6} + \text{etc.}\right)$$

$$\Pi : x = \frac{1}{2\sqrt{2}}\left(\frac{1}{3} + \frac{1}{2\cdot 7}\cdot\frac{1}{2^2} + \frac{1\cdot 3}{2\cdot 4\cdot 11}\cdot\frac{1}{2^4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 15}\cdot\frac{1}{2^6} + \text{etc.}\right).$$

Simili vero modo erunt

$$\Theta : y = \frac{1}{\sqrt{3}}\left(1 + \frac{1}{2\cdot 5}\cdot\frac{1}{3^2} + \frac{1\cdot 3}{2\cdot 4\cdot 9}\cdot\frac{1}{3^4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 13}\cdot\frac{1}{3^6} + \text{etc.}\right)$$

$$\Pi : y = \frac{1}{3\sqrt{3}}\left(\frac{1}{3} + \frac{1}{2\cdot 7}\cdot\frac{1}{3^2} + \frac{1\cdot 3}{2\cdot 4\cdot 11}\cdot\frac{1}{3^4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 15}\cdot\frac{1}{3^6} + \text{etc.}\right).$$

39. Hae series manifesto tantopere convergunt, ut, qui laborem calculi suscipere voluerit, veros litterarum a et c valores tam exacte definire queat, quam lubuerit; valores autem, quos supra assignavimus, iam tam parum a veritate discrepant, ut pro nostro instituto abunde sufficere possint; quandoquidem hic de eo tantum agitur, ut valores inventi calculum subducendo comprobari queant; quamobrem ad alias insignes proprietates huius curvae progrediamur.

PROBLEMA 3

40. *Proposito in curva elastica arcu quocunque PQ* (Fig. 5) *a puncto dato R abscindere arcum RS, qui illi arcui PQ sit aequalis.*

SOLUTIO

Quoniam igitur in curva quatuor puncta P, Q, R, S consideranda veniunt, sint abscissae illis respondentes $Cp = p$, $Cq = q$, $Cr = r$, $Cs = s$, pro quibus ponamus brevitatis gratia formulas irrationales

$$\sqrt{(1-p^4)} = P, \quad \sqrt{(1-q^4)} = Q, \quad \sqrt{(1-r^4)} = R$$

et

$$\sqrt{(1-s^4)} = S.$$

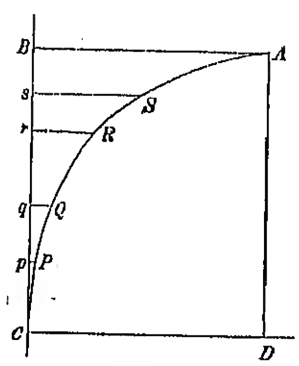

Fig. 5.

His positis, quoniam arcus RS aequalis esse debet arcui PQ, requiri[illegible]

$$CS - CR = CQ - CP,$$

hoc est

$$\Theta : s - \Theta : r = \Theta : q - \Theta : p;$$

cui aequationi ut per regulam supra datam satisfaciamus, quaeram[illegible] $\Theta : v$, ut sit

$$\Theta : v = \Theta : q - \Theta : p,$$

et secundum praecepta superiora esse debet

$$v = \frac{qP - pQ}{1 + ppqq},$$

unde fit

$$\sqrt{(1 - v^4)} = V = \frac{(1 - ppqq)PQ + 2pq(pp + qq)}{(1 + ppqq)^2}.$$

Hoc iam arcu invento esse debet $\Theta : s = \Theta : r + \Theta : v$; quare per [illegible]cepta fiet

$$s = \frac{rV + vR}{1 + rrvv}$$

hincque porro

$$S = \frac{(1 - rrvv)RV - 2rv(rr + vv)}{(1 + rrvv)^2}.$$

Substituamus nunc in his formulis valores pro v et V inventos; ac p[illegible]

$$1 + rrvv = \frac{(1 + ppqq)^2 + rrppQQ + rrqqPP - 2rrpqPQ}{(1 + ppqq)^2},$$

quae aequatio, si loco PP et QQ valores substituantur, ad hanc [illegible]

$$1 + rrvv = \frac{(1 + ppqq)^2 + rr(pp + qq)(1 - ppqq) - 2pqrrPQ}{(1 + ppqq)^2}.$$

At vero pro numeratore erit

$$rV + vR = \frac{r(1 - ppqq)PQ + 2pqr(pp + qq) + (qPR - pQR)(1 + pp\text{[illegible]}}{(1 + ppqq)^2}$$

consequenter abscissa quaesita $CS = s$ ita erit expressa

$$s = \frac{r(1-ppqq)PQ + 2pqr(pp+qq) + (qPR - pQR)(1+ppqq)}{(1+ppqq)^2 + rr(pp+qq)(1-ppqq) - 2pqrrPQ}.$$

Quod autem ad valorem litterae S attinet, quia eo in nostro calculo non indigemus, eius evolutione supersedemus.

41. Hinc igitur videmus, quomodo abscissa s per ternas abscissas datas p, q et r exprimatur; ubi quidem plurimum abest, ut litterae p, q, r in eam aequaliter ingrediantur, cum tamen ex aequatione proposita

$$\Theta : s = \Theta : r + \Theta : q - \Theta : p$$

intelligatur istas litteras p, q et r simili modo in valorem ipsius s ingredi debere, si modo littera p negative acciperetur. Neque igitur ullum est dubium, quin forma inventa ita transformari possit, ut ista paritas litterarum p, q et r elucescat, id quod tamen neutiquam liquet.

42. Cum autem esse debeat $\Theta : s = \Theta : r + \Theta : q - \Theta : p$, evidens est manente littera p binas reliquas q et r inter se commutari posse, unde etiam vera esse debet ista expressio

$$s = \frac{q(1-pprr)PR + 2pqr(pp+rr) + (rPQ - pQR)(1+pprr)}{(1+pprr)^2 + qq(pp+rr)(1-pprr) - 2prqqPR}.$$

Deinde manente r litterae p et q ita permutari poterunt, si loco q scribatur $-p$ et $-q$ loco p; tum autem erit

$$s = \frac{-p(1-qqrr)QR + 2pqr(qq+rr) + (qPR + rPQ)(1+qqrr)}{(1+qqrr)^2 + pp(qq+rr)(1-qqrr) + 2qrppQR}.$$

Atque hae tres expressiones, quantumvis diversae videantur, tamen certe eundem valorem exprimunt.

43. Insignis igitur hic occurrit quaestio analytica, quomodo istae tres expressiones tractari debeant, ut perfecta permutabilitas inter ternas litteras p, q, r perspiciatur. Facile quidem intelligitur, si tres istae expressiones in se invicem multiplicentur, ita ut productum aequetur cubo s^3, tum tam in numeratore quam in denominatore ternas litteras p, q, r pari modo esse ingressuras; verum tale productum nimis foret perplexum, quam ut ullum usum habere posset.

SCHOLION

44. Quae hactenus de curva elastica rectangula sunt tradita, etiam ad omnes curvas elasticas in genere accommodari poterunt. Cum enim pro data abscissa z sit applicata

$$=\int\frac{dz(\alpha+\beta zz)}{\sqrt{(1-(\alpha+\beta zz)^2)}}$$

et ipse arcus

$$=\int\frac{dz}{\sqrt{(1-(\alpha+\beta zz)^2)}},$$

praecepta generalia supra tradita pro comparatione harum quantitatum transcendentium simili modo applicari poterunt. Interim tamen hic conditio maxime necessaria probe notari debet, qua postulatur, ut denominator, qui evolutus est $\sqrt{(1-\alpha\alpha-2\alpha\beta zz-\beta\beta z^4)}$, ad hanc formam $\sqrt{(1+mzz+nz^4)}$ reduci queat, quod manifesto fieri nequit, nisi $1-\alpha\alpha$ fuerit quantitas positiva. His igitur casibus $\alpha\alpha<1$ omnes comparationes, quas tam inter arcus quam inter applicatas docuimus, simili modo ad curvas elasticas obliquangulas traduci poterunt.

DE SUPERFICIE CONI SCALENI UBI IMPRIMIS INGENTES DIFFICULTATES QUAE IN HAC INVESTIGATIONE OCCURRUNT PERPENDUNTUR

Convent. exhib. die 12 Septembris 1776

Commentatio 624 indicis ENESTROEMIANI

Nova acta academiae scientiarum Petropolitanae 3 (1785), 1788, p. 69—89

Summarium ibidem p. 173—175

SUMMARIUM

Le titre de ce mémoire annonce assez clairement ce qu'on y doit attendre: une exposition des difficultés, dont ce sujet, traité avec peu de succès par plusieurs Géomètres, est enveloppé, plutôt qu'une solution complette et satisfaisante de ce problème. En nommant la hauteur du cone a, son obliquité, ou bien la distance du centre à la perpendiculaire tirée du sommet sur le plan prolongé de la base $= b$, le rayon de la base $= c$ et la surface d'une portion infiniment-petite de la surface du cone comprise entre un arc de la base $c\partial\varphi$ et les deux côtés du cone $= \partial S$ cette surface est exprimée ainsi

$$\partial S = \frac{1}{2} c\partial\varphi \sqrt{aa + (c + b \cos.\varphi)^2},$$

comme on sait par le mémoire de feu M. EULER *De superficie conorum scalenorum aliorumque corporum conicorum*[1]), qui se trouve dans le premier volume des Nouveaux Commentaires, où le problème est réduit à la même expression. Mais ayant réduit alors la surface du cone à la rectification d'une courbe algébrique du sixième degré, il employe ici la voye

1) L. EULERI Commentatio 133 (indicis ENESTROEMIANI), Novi comment. acad. sc. Petrop. 1 (1747/8), 1750, p. 3—19; *LEONHARDI EULERI Opera omnia*, series I, vol. 27. A. K.

de l'approximation, en transformant l'expression irrationelle en série. Il met, pour cet effet,

$$aa + \frac{1}{2}bb + cc = ff \quad \text{et} \quad 2bc \cos.\varphi + \frac{1}{2}bb \cos. 2\varphi = v,$$

de façon que

$$\partial S = \frac{1}{2}c\partial\varphi\sqrt{ff+v} \quad \text{et} \quad \sqrt{ff+v} = f + \frac{1}{2}\cdot\frac{v}{f} - \frac{1\cdot 1}{2\cdot 4}\cdot\frac{vv}{f^3} + \text{etc.},$$

et il assigne la valeur de la surface S pour les deux, trois et quatre premiers termes de cette série, où la dernière expression, composée des quatre premiers termes, est assez approchante, pourvu que f soit considérablement plus grand que b et c.

Une autre approximation déduite de la transformation du radical

$$\sqrt{aa + (c + b\cos.\varphi)^2}$$

en série donne une loi de progression plus manifeste. L'Auteur ne la pousse cependant que jusqu'à la somme de quatre termes; mais il fait voir, comment on peut la pousser plus loin, et il donne à la fin de son mémoire cette expression pour la surface entière du cone $\pi aaxu\cdot V$, où $x = \frac{c}{a}$ et $u = \sqrt{1+xx}$ et V une série dont la loi de progression est évidente, mais qui n'est d'aucun usage, lorsque l'obliquité du cone n'est pas très-petite en comparaison de la hauteur du cone et du rayon de sa base.

Une grande difficulté se présente lorsqu'on cherche la surface d'un cone oblique dont la hauteur est très-petite. Car alors la série qui exprime le radical

$$\sqrt{aa + (c + b\cos.\varphi)^2},$$

devient

$$c + b\cos.\varphi + \frac{1}{2}\cdot\frac{aa}{c + b\cos.\varphi} - \frac{1\cdot 1}{2\cdot 4}\cdot\frac{a^4}{(c + b\cos.\varphi)^3} + \text{etc}$$

et cette série est très-convergente, lorsque la hauteur a est très-petite par rapport à $c + b\cos.\varphi$; mais comme parmi les valeurs de l'angle φ il y en a où $\cos.\varphi = -\frac{c}{b}$ et partant $c + b\cos.\varphi = 0$, tous les termes après le premier deviennent infiniment grands et s'écartent par conséquent énormément de la vérité, inconvénient que l'Analyse n'a pas encore réussi à lever. Dans tous ces cas il faudra donc recourir à la dimension pratique, en partageant toute la surface du cone en plusieurs parties, et chercher la surface de chacune séparément.

Pour faciliter cette opération l'Auteur cherche la figure qui nait du développement de la surface du cone en surface plane, ce qui le mène à une courbe transcendante qu'on ne peut exprimer ni par des logarithmes, ni par des arcs de cercle, mais dont néanmoins M. Euler est en état d'assigner quelques propriétés remarquables. D'ailleurs comme elle peut être représentée par le développement d'un papier appliqué à la surface du cone, elle fournit un nouvel exemple d'une courbe hyperscendante dont la construction mécanique est très-facile.

1. Sit circulus $EGFH$ (Fig. 1) basis coni scaleni propositi, cuius vertex in sublimi situs sit A, unde ad planum basis demittatur perpendiculum AB, et ex B per centrum basis C agatur recta $BFCE$. Vocetur altitudo $AB = a$, deinde vero sit $BC = b$, quae linea exhibet coni obliquitatem; si enim esset $b = 0$, conus foret rectus. Denique vero vocetur radius basis $CE = CF = c$ ac manifestum est his tribus quantitatibus a, b, c naturam coni penitus determinari. Hinc si ad verticem ductae intelligantur rectae EA et FA, ob $BE = b + c$ et $BF = b - c$ erit

$$AE = \sqrt{aa + (b + c)^2},$$

quod est latus coni maximum; latus vero minimum erit

$$AF = \sqrt{aa + (b - c)^2}.$$

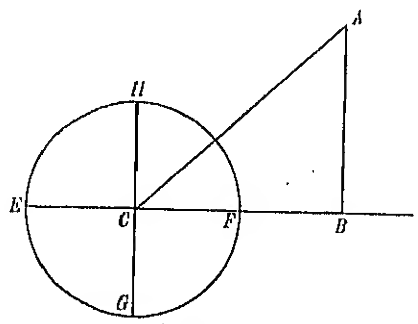

Fig. 1.

Praeterea si in basi ducatur diameter GH ad EF normalis, rectae AG et AH erunt latera media coni inter se aequalia; ad quorum quantitatem inveniendam, quoniam est $AC = \sqrt{(aa + bb)}$ et triangula ACG et ACH ad C rectangula, erit

$$AG = AH = \sqrt{(aa + bb + cc)}.$$

2. Quoniam igitur nobis propositum est superficiem huius coni scaleni indagare, quemadmodum ea scilicet per terna elementa a, b et c definiatur, haec investigatio facillime sequenti modo instituetur. Ducto coni latere maximo AE (Fig. 2) in basi coni ex centro C capiatur angulus indefinitus $ECS = \varphi$, qui suo differentiali $SCs = \partial\varphi$ augeatur, ac vocetur portio superficiei conicae inter rectas AE et AS atque arcum ES inclusa $= S$, ita ut posito $\varphi = 180°$ punctum S in F perveniat et ista quantitas S nobis sit indicatura semissem superficiei conicae eiusque ergo duplum totam superficiem coni quaesitam. Quodsi iam ex A ducamus rectam proximam As,

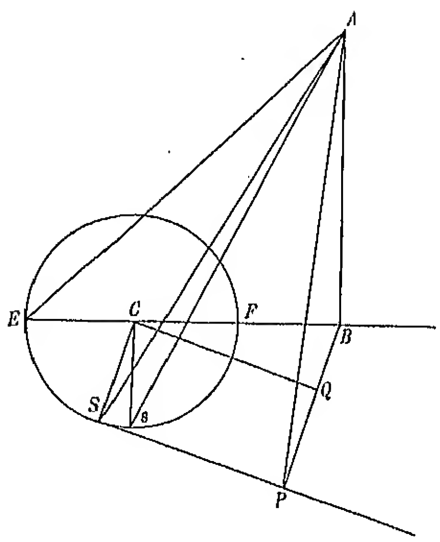

Fig. 2.

area trianguli elementaris SAs dabit valorem differentialis ∂S, ita ut totum negotium huc redeat, ut area istius trianguli SAs exploretur, quod ob arculum $Ss = c\partial\varphi$ ideoque infinite parvum tanquam triangulum rectilineum spectari potest.

3. Hunc in finem ducatur ad S tangens circuli SP sive, quod eodem redit, producatur elementum Ss, ita ut recta SP sit basis Ss producta; unde, si ex A ad eam ducatur perpendicularis AP, erit area trianguli ASs sive ∂S

$$= \frac{Ss \cdot AP}{2} = \frac{1}{2} AP \cdot c\,\partial\varphi.$$

Constat autem hoc perpendiculum AP duci, si ex puncto B ad rectam SP demittatur perpendiculum BP, quandoquidem tum etiam recta AP ei erit normalis. Iam ex C ad rectam BP normaliter agatur recta CQ, et quia BP parallela est radio CS, erit angulus $CBQ = \varphi$, unde ob $BC = b$ erit $CQ = b \sin.\varphi$ et $BQ = b \cos.\varphi$. Quare, cum sit $PQ = CS = c$, erit

$$BP = c + b \cos.\varphi$$

et intervallum $SP = CQ = b \sin.\varphi$ ideoque ex triangulo APB, quia AB ad BP est perpendicularis, reperietur hypothenusa

$$AP = \sqrt{aa + (c + b \cos.\varphi)^2};$$

consequenter hinc elicimus elementum superficiei quaesitum

$$\partial S = \frac{1}{2} c\,\partial\varphi \sqrt{aa + (c + b \cos.\varphi)^2}.$$

Sicque tota investigatio huc est perducta, ut ista formula differentialis integretur.

4. Consideremus primo casum coni recti, qui prodit facta obliquitate $b = 0$. Hoc ergo casu habebimus $\partial S = \frac{1}{2} c\,\partial\varphi \sqrt{(aa + cc)}$, unde integrando fit $S = \frac{1}{2} c\varphi \sqrt{(aa + cc)}$. Fiat nunc $\varphi = 180^\circ$ sive $\varphi = \pi$ et semissis superficiei conicae erit $= \frac{1}{2}\pi c \sqrt{(aa + cc)}$ ideoque tota coni superficies

$$= \pi c \sqrt{(aa + cc)};$$

ubi notetur formulam $\sqrt{(aa+cc)}$ exprimere latus huius coni recti, tum vero totam basis peripheriam esse $=2\pi c$. Constat autem superficiem coni recti inveniri, si latus coni ducatur in dimidiam basis circumferentiam.

5. Hinc autem facile intelligitur pro conis scalenis hanc investigationem multo magis fieri arduam, propterea quod ea pendet ab integratione huius formulae

$$\partial S=\frac{1}{2}c\partial\varphi\sqrt{aa+(c+b\cos.\varphi)^2},$$

quae evoluta praebet

$$\partial S=\frac{1}{2}c\partial\varphi\sqrt{aa+cc+2bc\cos.\varphi+bb\cos.\varphi^2},$$

quae ob $\cos.\varphi^2=\frac{1}{2}+\frac{1}{2}\cos.2\varphi$ etiam transmutari potest in hanc formam

$$\partial S=\frac{1}{2}c\partial\varphi\sqrt{\left(aa+\frac{1}{2}bb+cc+2bc\cos.\varphi+\frac{1}{2}bb\cos.2\varphi\right)}.$$

Huius autem formulae integratio absoluta nullo modo sperari potest, siquidem certum est eam neque per logarithmos neque per arcus circulares expediri posse; quamobrem nobis tantum in approximationibus erit acquiescendum.

6. Ponamus brevitatis gratia

$$aa+\frac{1}{2}bb+cc=ff,$$

ut habeamus

$$\partial S=\frac{1}{2}c\partial\varphi\sqrt{ff+2bc\cos.\varphi+\frac{1}{2}bb\cos.2\varphi},$$

ubi primo observandum occurrit, si quantitas ff fuerit valde magna prae binis reliquis terminis, tum approximationem nullam moram facessere; si enim ponamus

$$2bc\cos.\varphi+\frac{1}{2}bb\cos.2\varphi=v,$$

ut sit

$$\partial S=\frac{1}{2}c\partial\varphi\sqrt{(ff+v)},$$

facta evolutione erit

$$\sqrt{(ff+v)}=f+\frac{1}{2}\cdot\frac{v}{f}-\frac{1\cdot1}{2\cdot4}\cdot\frac{vv}{f^3}+\frac{1\cdot1\cdot3}{2\cdot4\cdot6}\cdot\frac{v^3}{f^5}-\frac{1\cdot1\cdot3\cdot5}{2\cdot4\cdot6\cdot8}\cdot\frac{v^4}{f^7}+\text{etc.},$$

quae series eo magis convergit, quo minor erit quantitas v prae ff; unde sufficiet huius seriei vel tantum binos terminos priores accipere vel insuper tertium vel adeo etiam quartum pluresve admittere, unde aliquot casus evolvamus.

CASUS 1
QUO APPROXIMATIO IN SECUNDO TERMINO SUBSISTIT

7. Hoc igitur casu habebimus

$$\partial S = \frac{1}{2} c\partial\varphi \left(f + \frac{v}{2f}\right),$$

ubi primus terminus integratus dat $\frac{1}{2} fc\varphi$, secundus vero terminus ob

$$v = 2bc \cos.\varphi + \frac{1}{2} bb \cos. 2\varphi$$

integratus praebet

$$\frac{c}{4f}\int \partial\varphi \left(2bc \cos.\varphi + \frac{1}{2} bb \cos. 2\varphi\right) = \frac{c}{4f}\left(2bc \sin.\varphi + \frac{1}{4} bb \sin. 2\varphi\right),$$

ita ut iam sit

$$S = \frac{1}{2} cf\varphi + \frac{bcc \sin.\varphi}{2f} + \frac{bbc \sin. 2\varphi}{16f}.$$

Fiat nunc $\varphi = \pi$ ac formula duplicata dabit totam coni superficiem $= \pi cf$, quae restituto pro f valore erit

$$S = \pi c \sqrt{\left(aa + \frac{1}{2} bb + cc\right)},$$

quae ergo sufficere potest, quoties quantitates $2bc$ et $\frac{1}{2} bb$ fuerint quam minimae respectu quantitatis $aa + \frac{1}{2} bb + cc$. Haec conditio imprimis locum habet, quando altitudo coni fuerit permagna prae obliquitate b atque etiam radio basis c. Ante autem vidimus, si obliquitas coni prorsus evanesceret, superficiem coni recti esse $= \pi c \sqrt{(aa + cc)}$; nunc igitur superficies tantillo est maior in ratione

$$\sqrt{(aa + cc)} : \sqrt{\left(aa + \frac{1}{2} bb + cc\right)}.$$

CASUS 2
QUO APPROXIMATIO IN TERTIO TERMINO SUBSISTIT

8. Quoniam hic tantum superficiem coni quaerimus, statim ponere possumus $S = c\int\partial\varphi\sqrt{(ff+v)}$; tum enim integratione peracta tantum opus est facere $\varphi = \pi$. Praesenti igitur casu erit

$$\partial S = c\partial\varphi\left(f + \frac{v}{2f} - \frac{vv}{8f^3}\right);$$

modo autem vidimus binos terminos priores dare πcf, ita ut sit

$$S = \pi cf - \frac{c}{8f^3}\int vv\partial\varphi.$$

Est vero

$$vv = 4bbcc\cos.\varphi^2 + 2b^3c\cos.\varphi\cos.2\varphi + \frac{1}{4}b^4\cos.2\varphi^2,$$

quae formula ob

$$\cos.\varphi^2 = \frac{1}{2} + \frac{1}{2}\cos.2\varphi,$$

$$\cos.\varphi\cos.2\varphi = \frac{1}{2}\cos.\varphi + \frac{1}{2}\cos.3\varphi$$

et

$$\cos.2\varphi^2 = \frac{1}{2} + \frac{1}{2}\cos.4\varphi,$$

transformatur in hanc

$$vv = 2bbcc + \frac{1}{8}b^4 + b^3c\cos.\varphi + 2bbcc\cos.2\varphi + b^3c\cos.3\varphi + \frac{1}{8}b^4\cos.4\varphi,$$

quae ergo formula constat quinque membris, quorum primum tantum in integratione est considerandum, propterea quod sequentes termini integrati darent $\sin.\varphi$, $\sin.2\varphi$, $\sin.3\varphi$ et $\sin.4\varphi$, qui posito $\varphi = \pi$ omnes in nihilum abeunt, ita ut pro hoc casu sit

$$\int vv\partial\varphi = \pi\left(2bbcc + \frac{1}{8}b^4\right);$$

quamobrem tota coni superficies erit

$$S = \pi cf - \frac{\pi bbc^3}{4f^3} - \frac{\pi b^4c}{64f^3},$$

quae formula iam multo propius ad veritatem accedit quam ea, quae casu primo est inventa.

CASUS 3
QUO APPROXIMATIO IN QUARTO TERMINO SISTITUR

9. Hic igitur ad expressionem modo inventam insuper adiici debet valor, qui ex hac formula integrali resultat

$$\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\cdot\frac{c}{f^5}\int v^3\partial\varphi,$$

postquam scilicet positum fuerit $\varphi=\pi$. Modo autem vidimus esse

$$vv=2bbcc+\frac{1}{8}b^4+b^3c\cos.\varphi+2bbcc\cos.2\varphi+b^3c\cos.3\varphi+\frac{1}{8}b^4\cos.4\varphi,$$

quae forma per

$$v=2bc\cos.\varphi+\frac{1}{2}bb\cos.2\varphi$$

multiplicata retentis tantum terminis constantibus, qui facta reductione superent, dabit

$$v^3=b^4cc+\frac{1}{2}b^4cc=\frac{3}{2}b^4cc,$$

unde fit $\int v^3\partial\varphi=\frac{3}{2}\pi b^4cc$, ita ut pars adiicienda sit $\frac{3\pi b^4c^3}{32f^5}$; consequenter adiecta etiam hac parte habebimus accuratius

$$S=\pi cf-\frac{\pi bbc^3}{4f^3}-\frac{\pi b^4c}{64f^3}+\frac{3\pi b^4c^3}{32f^5}.$$

10. Contemplemur hic casum, quo obliquitas b ipsi radio baseos est aequalis sive ubi perpendiculum AB in ipsum punctum F incidit. Facto igitur $b=c$ superficies huius coni, dum approximatio usque ad quartum terminum producitur, erit

$$S=\pi cf-\frac{17\pi c^5}{64f^3}+\frac{3\pi c^7}{32f^5}$$

sive

$$S=\pi cf\left(1-\frac{17c^4}{64f^4}+\frac{3c^6}{32f^6}\right),$$

ubi notetur esse $ff=aa+\frac{3}{2}cc$. Haec expressio eo propius ad veritatem accedit, quo maior fuerit quantitas f prae radio basis c. Ita si altitudo coni

diametro baseos aequetur, ita ut sit $ff = \frac{11}{2}cc$, tum superficies huius coni erit

$$S = \pi c^2 \sqrt{\frac{11}{2}} \cdot \left(1 - \frac{17}{16 \cdot 121} + \frac{3}{4 \cdot 1331}\right),$$

quae partes in unam contractae praebent superficiem coni

$$S = \frac{21121}{21296} \pi c^2 \sqrt{\frac{11}{2}}.$$

11. Hanc autem approximandi methodum non ad plures terminos prosequimur, quoniam calculus nimis fieret molestus neque ulla lex progressionis perspici posset. Plerumque autem approximatio postrema sufficere posse videtur, dummodo quantitas f notabiliter superet ambas quantitates b et c. Tentemus autem aliam methodum, quae quidem pariter postulat, ut altitudo coni a plurimum superet bina reliqua elementa b et c, quae autem quandam legem progressionis pollicetur, ita ut approximationem pro lubitu continuo ulterius persequi liceat.

ALIA METHODUS APPROXIMANDI QUANDO a MULTUM SUPERAT b ET c

12. Hic scilicet formulam $(c + b \cos.\varphi)^2$ non evolvemus, sed cum sit per seriem

$$\sqrt{aa + (c + b\cos.\varphi)^2} = a + \frac{1}{2} \cdot \frac{(c+b\cos.\varphi)^2}{a} - \frac{1 \cdot 1}{2 \cdot 4} \cdot \frac{(c+b\cos.\varphi)^4}{a^3}$$
$$+ \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} \cdot \frac{(c+b\cos.\varphi)^6}{a^5} - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{(c+b\cos.\varphi)^8}{a^7} + \text{etc.},$$

singulas potestates pares ipsius $c + b \cos.\varphi$ ita evolvamus, ut statim omnes potestates ipsius $\cos.\varphi$ ad cosinus simplices revocemus; tum enim omnia membra per quempiam cosinum affecta tuto reiicere poterimus, propterea quod in integratione praebent sinus angulorum multiplorum ipsius φ, qui posito $\varphi = \pi$ omnes in nihilum essent abituri.

13. Quo igitur hoc negotium facilius expediri queat, ante omnia observasse iuvabit omnes potestates impares ipsius $\cos.\varphi$ nullam suppeditare

quantitatem absolutam, ita ut has potestates penitus omittere liceat; ex potestatibus autem paribus sequentes nascuntur quantitates absolutae

$$\cos.\varphi^2 = \frac{1}{2}$$
$$\cos.\varphi^4 = \frac{1\cdot 3}{2\cdot 4}$$
$$\cos.\varphi^6 = \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}$$
$$\cos.\varphi^8 = \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8} \quad \text{etc.}$$

Iuxta hanc igitur regulam potestates pares binomii $c + b\cos.\varphi$ evolvamus eritque

$$(c + b\cos.\varphi)^2 = cc + \frac{1}{2}bb$$
$$(c + b\cos.\varphi)^4 = c^4 + 3bbcc + \frac{1\cdot 3}{2\cdot 4}b^4$$
$$(c + b\cos.\varphi)^6 = c^6 + \frac{15}{2}bbc^4 + \frac{1\cdot 3\cdot 15}{2\cdot 4}b^4cc + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}b^6.$$

Quin etiam res in genere hoc modo expedietur

$$\begin{aligned}(c + b\cos.\varphi)^{2n} = c^{2n} &+ \frac{2n}{1}\cdot\frac{2n-1}{2}\cdot\frac{1}{2}b^2c^{2n-2}\\ &+ \frac{2n}{1}\cdot\frac{2n-1}{2}\cdot\frac{2n-2}{3}\cdot\frac{2n-3}{4}\cdot\frac{1\cdot 3}{2\cdot 4}b^4c^{2n-4}\\ &+ \frac{2n}{1}\cdot\frac{2n-1}{2}\cdot\frac{2n-2}{3}\cdot\frac{2n-3}{4}\cdot\frac{2n-4}{5}\cdot\frac{2n-5}{6}\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}b^6c^{2n-6} + \text{etc.}\end{aligned}$$

14. Introducamus nunc istos valores in seriem pro

$$\sqrt{aa + (c + b\cos.\varphi)^2}$$

exhibitam et statim per πc multiplicemus atque integra coni superficies sequenti modo exprimetur

$$\begin{aligned}S = \pi ca &+ \frac{1\pi c}{2a}\left(cc + \frac{1}{2}bb\right) - \frac{1\cdot 1\pi c}{2\cdot 4a^3}\left(c^4 + 3bbcc + \frac{1\cdot 3}{2\cdot 4}b^4\right)\\ &+ \frac{1\cdot 1\cdot 3\pi c}{2\cdot 4\cdot 6a^5}\left(c^6 + \frac{15}{2}bbc^4 + \frac{1\cdot 3\cdot 15}{2\cdot 4}b^4cc + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}b^6\right)\\ &- \frac{1\cdot 1\cdot 3\cdot 5\pi c}{2\cdot 4\cdot 6\cdot 8a^7}\left(\begin{aligned}&c^8 + \frac{8\cdot 7\cdot 1}{1\cdot 2\cdot 2}bbc^6 + \frac{8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{1\cdot 3}{2\cdot 4}b^4c^4\\ &+ \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}b^6cc + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}b^8\end{aligned}\right) + \text{etc.}\end{aligned}$$

15. Quodsi ex singulis membris terminos tantum primos excerpamus, ii constituent hanc seriem

$$\pi c\left(a+\frac{1cc}{2a}-\frac{1\cdot 1}{2\cdot 4}\cdot\frac{c^4}{a^3}+\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\cdot\frac{c^6}{a^5}-\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{c^8}{a^7}+\text{etc.}\right),$$

quae series manifesto convenit cum ea, quam formula $\sqrt{(aa+cc)}$ producit, quamobrem loco omnium terminorum primorum scribere licebit

$$\pi c\sqrt{(aa+cc)}.$$

Simili modo secundos terminos singulorum membrorum excerpamus, qui dabunt hanc seriem

$$\frac{\pi bbc}{2}\left(\frac{1}{2a}-\frac{1\cdot 1\cdot 6cc}{2\cdot 4a^3}+\frac{1\cdot 1\cdot 3\cdot 15c^4}{2\cdot 4\cdot 6a^5}-\frac{1\cdot 1\cdot 3\cdot 5\cdot 28c^6}{2\cdot 4\cdot 6\cdot 8a^7}+\text{etc.}\right)$$

sive

$$\frac{\pi bbc}{2a}\left(\frac{1}{2}-\frac{1\cdot 1}{2\cdot 4}\cdot\frac{4\cdot 3}{1\cdot 2}\cdot\frac{cc}{aa}+\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\cdot\frac{6\cdot 5}{1\cdot 2}\cdot\frac{c^4}{a^4}-\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{8\cdot 7}{1\cdot 2}\cdot\frac{c^6}{a^6}+\text{etc.}\right),$$

quae etiam hoc modo repraesentari potest

$$\frac{\pi bbc}{4a}\left(1-\frac{3}{2}\cdot\frac{cc}{aa}+\frac{3\cdot 5}{2\cdot 4}\cdot\frac{c^4}{a^4}-\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\cdot\frac{c^6}{a^6}+\frac{3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{c^8}{a^8}+\text{etc.}\right),$$

cuius seriei valor manifesto est $\left(1+\frac{cc}{aa}\right)^{-\frac{3}{2}}$, ita ut summa omnium terminorum secundorum sit

$$=\frac{\pi aabbc}{4(aa+cc)^{\frac{3}{2}}}.$$

16. Colligamus eodem modo omnia tertia membra singulorum terminorum, qui omnes affecti sunt potestate b^4 et constituunt hanc seriem

$$-\frac{1\cdot 1}{2\cdot 4}\pi c\cdot\frac{1\cdot 3}{2\cdot 4}\cdot\frac{b^4}{a^3}+\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\pi c\cdot\frac{1\cdot 3}{2\cdot 4}\cdot\frac{6\cdot 5\cdot 4\cdot 3}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{b^4cc}{a^5}$$
$$-\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\pi c\cdot\frac{1\cdot 3}{2\cdot 4}\cdot\frac{8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{b^4c^4}{a^7}+\frac{1\cdot 1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}\pi c\cdot\frac{1\cdot 3}{2\cdot 4}\cdot\frac{10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{b^4c^6}{a^9}+\text{etc.},$$

qui termini reducuntur ad sequentem expressionem

$$-\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi b^4c}{24a^3}\left(3\cdot 1-\frac{3\cdot 5\cdot 3}{2}\cdot\frac{cc}{aa}+\frac{3\cdot 5\cdot 7\cdot 5}{2\cdot 4}\cdot\frac{c^4}{a^4}-\frac{3\cdot 5\cdot 7\cdot 9\cdot 7}{2\cdot 4\cdot 6}\cdot\frac{c^6}{a^6}+\text{etc.}\right).$$

17. Ista series sequenti modo in clariorem ordinem redigi poterit

$$-\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi b^4 c}{8a^3}\left(1-\frac{5\cdot 3}{2}\cdot\frac{cc}{aa}+\frac{5\cdot 7\cdot 5}{2\cdot 4}\cdot\frac{c^4}{a^4}-\frac{5\cdot 7\cdot 9\cdot 7}{2\cdot 4\cdot 6}\cdot\frac{c^6}{a^6}+\text{etc.}\right).$$

Ponamus hic brevitatis gratia $\frac{cc}{aa}=xx$ atque factorem communem $-\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi b^4 c}{8a^3}$ multiplicari oportebit per hanc seriem

$$s=1-\frac{5\cdot 3}{2}xx+\frac{5\cdot 7\cdot 5}{2\cdot 4}x^4-\frac{5\cdot 7\cdot 9\cdot 7}{2\cdot 4\cdot 6}x^6+\frac{5\cdot 7\cdot 9\cdot 11\cdot 9}{2\cdot 4\cdot 6\cdot 8}x^8-\text{etc.}$$

Haec series iam satis est regularis, et nisi postremi factores numerici adessent, eius summatio in promptu foret. Ad hos igitur factores tollendos utamur integratione ac reperiemus

$$\int s\,\partial x=x-\frac{5}{2}x^3+\frac{5\cdot 7}{2\cdot 4}x^5-\frac{5\cdot 7\cdot 9}{2\cdot 4\cdot 6}x^7+\frac{5\cdot 7\cdot 9\cdot 11}{2\cdot 4\cdot 6\cdot 8}x^9-\text{etc.}$$

Novimus autem esse

$$(1+xx)^{-\frac{5}{2}}=1-\frac{5}{2}xx+\frac{5\cdot 7}{2\cdot 4}x^4-\frac{5\cdot 7\cdot 9}{2\cdot 4\cdot 6}x^6+\text{etc.},$$

unde patet fore

$$\int s\,\partial x=x(1+xx)^{-\frac{5}{2}},$$

hincque differentiando colligitur

$$s=(1+xx)^{-\frac{5}{2}}-5xx(1+xx)^{-\frac{7}{2}}$$

sive

$$s=\frac{1-4xx}{(1+xx)^{\frac{7}{2}}}.$$

18. Restituamus nunc loco xx valorem $\frac{cc}{aa}$ fietque

$$s=\frac{a^5(aa-4cc)}{(aa+cc)^{\frac{7}{2}}},$$

qui valor multiplicatus per factorem communem $-\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi b^4 c}{8a^3}$ dabit summam omnium terminorum tertiorum, quae ergo erit

$$=-\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi aab^4 c}{8}\cdot\frac{aa-4cc}{(aa+cc)^{\frac{7}{2}}};$$

quamobrem si istae summae terminorum primorum, secundorum ac tertiorum coniungantur, pro superficie nostri coni scaleni nanciscemur sequentem expressionem

$$S=\pi c\sqrt{(aa+cc)}+\frac{\pi aabbc}{4(aa+cc)^{\frac{3}{2}}}-\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi aab^4c(aa-4cc)}{8(aa+cc)^{\frac{7}{2}}},$$

ita ut tantum supersit insuper terminos quartos, quintos etc. investigare, quos autem plerumque negligere licebit. Facile autem intelligitur, si etiam hos terminos summare voluerimus, denominatores futuros esse

$$(aa+cc)^{\frac{11}{2}},\quad (aa+cc)^{\frac{15}{2}}\quad \text{etc.};$$

verum numeratores nimis operosum foret explorare.

19. Tentemus igitur summationem terminorum quartorum, qui adhibita simili operatione talem progressionem suppeditant, cuius factor communis est

$$\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{3\cdot 5}{1\cdot 2\cdots 6}\cdot\frac{\pi b^6c}{a^5}=\frac{1\cdot 3\cdot 5}{2^2\cdot 4^2\cdot 6^2}\cdot\frac{\pi b^6c}{a^5},$$

in quem duci debebit haec series

$$3-\frac{7\cdot 5\cdot 3}{2}\cdot\frac{c^2}{a^2}+\frac{7\cdot 9\cdot 7\cdot 5}{2\cdot 4}\cdot\frac{c^4}{a^4}-\frac{7\cdot 9\cdot 11\cdot 9\cdot 7}{2\cdot 4\cdot 6}\cdot\frac{c^6}{a^6}+\frac{7\cdot 9\cdot 11\cdot 13\cdot 11\cdot 9}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{c^8}{a^8}-\text{etc.}$$

Fiat igitur iterum $\frac{cc}{aa}=xx$ ac ponatur

$$s=3-\frac{7\cdot 5\cdot 3}{2}xx+\frac{7\cdot 9\cdot 7\cdot 5}{2\cdot 4}x^4-\frac{7\cdot 9\cdot 11\cdot 9\cdot 7}{2\cdot 4\cdot 6}x^6+\frac{7\cdot 9\cdot 11\cdot 13\cdot 11\cdot 9}{2\cdot 4\cdot 6\cdot 8}x^8-\text{etc.},$$

cuius ergo seriei summam indagari oportet, id quod sequenti modo sumus expedituri.

20. Primo scilicet, ut factores postremi tollantur, per integrationem formetur ista series

$$\int s\,\partial x=3x-\frac{7\cdot 5}{2}x^3+\frac{7\cdot 9\cdot 7}{2\cdot 4}x^5-\frac{7\cdot 9\cdot 11\cdot 9}{2\cdot 4\cdot 6}x^7+\frac{7\cdot 9\cdot 11\cdot 13\cdot 11}{2\cdot 4\cdot 6\cdot 8}x^9-\text{etc.}$$

Ut nunc hinc denuo ultimos factores tollamus, multiplicemus per $x\partial x$ et integrando reperiemus

$$\int x\partial x\int s\partial x = x^3 - \frac{7}{2}x^5 + \frac{7\cdot 9}{2\cdot 4}x^7 - \frac{7\cdot 9\cdot 11}{2\cdot 4\cdot 6}x^9 + \frac{7\cdot 9\cdot 11\cdot 13}{2\cdot 4\cdot 6\cdot 8}x^{11} - \text{etc.}$$

Cum igitur sit

$$(1+xx)^{-\frac{7}{2}} = 1 - \frac{7}{2}xx + \frac{7\cdot 9}{2\cdot 4}x^4 - \frac{7\cdot 9\cdot 11}{2\cdot 4\cdot 6}x^6 + \frac{7\cdot 9\cdot 11\cdot 13}{2\cdot 4\cdot 6\cdot 8}x^8 - \text{etc.},$$

manifestum est fore

$$\int x\partial x\int s\partial x = x^3(1+xx)^{-\frac{7}{2}},$$

cuius differentiale per $x\partial x$ divisum dabit

$$\int s\partial x = 3x(1+xx)^{-\frac{7}{2}} - 7x^3(1+xx)^{-\frac{9}{2}};$$

haecque formula denuo differentiata praebet

$$s = 3(1+xx)^{-\frac{7}{2}} - 42xx(1+xx)^{-\frac{9}{2}} + 63x^4(1+xx)^{-\frac{11}{2}},$$

quae expressio porro reducitur ad hanc

$$s = \frac{3 - 36xx + 24x^4}{(1+xx)^{\frac{11}{2}}}.$$

Scribendo igitur $\frac{cc}{aa}$ loco xx erit

$$s = \frac{a^7(3a^4 - 36aacc + 24c^4)}{(aa+cc)^{\frac{11}{2}}},$$

quae formula ducta in factorem communem $\frac{1\cdot 3\cdot 5}{2^2\cdot 4^2\cdot 6^2}\cdot\frac{\pi b^6 c}{a^5}$ praebet summam omnium terminorum quartorum

$$= \frac{1\cdot 3\cdot 5}{2^2\cdot 4^2\cdot 6^2}\cdot\frac{\pi aab^6c(3a^4 - 36aacc + 24c^4)}{(aa+cc)^{\frac{11}{2}}}.$$

21. Evolutio ista postrema nobis hoc eximium commodum praestat, ut etiam legem, qua sequentium terminorum summae progrediuntur, patefaciat. Quemadmodum enim, si summa terminorum tertiorum statuatur

$$= -\frac{1\cdot 3}{2^2\cdot 4^2}\cdot\frac{\pi b^4 c}{a^3}\cdot s,$$

posito $\frac{cc}{aa} = xx$ pro s pervenimus ad hanc aequationem

$$\int s\partial x = x(1+xx)^{-\frac{5}{2}},$$

ita pro terminis quartis, si earum summa ponatur

$$= \frac{1\cdot 3\cdot 5}{2^2\cdot 4^2\cdot 6^2}\cdot\frac{\pi b^6 c}{a^5}s,$$

pro s invenimus hanc aequationem

$$\int x\partial x \int s\partial x = x^3(1+xx)^{-\frac{7}{2}}.$$

Hoc modo facile patet, si summa terminorum quintorum ponatur

$$= -\frac{1\cdot 3\cdot 5\cdot 7}{2^2\cdot 4^2\cdot 6^2\cdot 8^2}\cdot\frac{\pi b^8 c}{a^7}s,$$

tum pro quantitate s invenienda prodituram esse hanc aequationem

$$\int x\partial x \int x\partial x \int s\partial x = x^5(1+xx)^{-\frac{9}{2}}.$$

Eodemque modo pro terminis sextis, si eorum summa statuatur

$$= \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2^2\cdot 4^2\cdot 6^2\cdot 8^2\cdot 10^2}\cdot\frac{\pi b^{10} c}{a^9}s,$$

tum quantitas s ex hac aequatione definiri debebit

$$\int x\partial x \int x\partial x \int x\partial x \int s\partial x = x^7(1+xx)^{-\frac{11}{2}}$$

sicque lex progressionis in infinitum penitus est manifesta.

22. Quoniam igitur summam terminorum quartorum nobis pariter evolvere licuit, eam insuper ad summam praecedentium addamus atque superficies nostri coni scaleni nunc accuratius sequenti forma exprimetur

$$\pi c\sqrt{(aa+cc)} + \frac{\pi aabbc}{2^2(aa+cc)^{\frac{3}{2}}} - \frac{1\cdot 3}{2^2\cdot 4^2}\cdot\frac{\pi aab^4c(aa-4cc)}{(aa+cc)^{\frac{7}{2}}}$$
$$+ \frac{1\cdot 3\cdot 5}{2^2\cdot 4^2\cdot 6^2}\cdot\frac{\pi aab^6c(3a^4-36aacc+24c^4)}{(aa+cc)^{\frac{11}{2}}},$$

quam formam semper adhibere licebit, quoties bb fuerit valde parvum prae $aa + cc$, id quod duplici modo contingere potest, vel quando altitudo coni a plurimum superat eius obliquitatem b, vel quando radius basis c multum excedit obliquitatem b; atque si haec utraque conditio locum habeat, ista formula eo magis ad veritatem appropinquabit.

23. Sin autem neutra harum conditionum locum inveniat atque obliquitas b tam ratione altitudinis a quam radii baseos c notabilem habeat magnitudinem vel adeo hos terminos superet, tum formula nostra inventa nullum plane usum praestare poterit. His igitur casibus maxima difficultas occurrit superficiem coni definiendi atque longe alia artificia desiderantur, quorum beneficio ista quaestio enodari queat.

24. Consideremus primo casum, quo altitudo coni a penitus evanescit, ita ut pro elemento superficiei habeamus hanc formulam

$$\partial S = c\partial\varphi \sqrt{(c + b\cos.\varphi)^2},$$

quam iam duplicavimus, ita ut integratione peracta tantum supersit statuere $\varphi = 180^\circ = \pi$. Cum igitur signum radicale quadrato sit praefixum, erit utique

$$\partial S = c\partial\varphi(c + b\cos.\varphi),$$

unde integrando elicitur $S = cc\varphi + bc\sin.\varphi$, unde facto $\varphi = 180^\circ$ tota superficies prodit $= \pi cc$ sicque ipsi areae basis erit aequalis, id quod per se est perspicuum, quoties vertex coni intra basin cadit; sin autem extra basin incidat, manifestum est superficiem coni multo maiorem fore quam aream baseos. Si enim talem conum charta obducere voluerimus, evidens est eo maius spatium requiri, quo longius vertex coni extra basin fuerit remotus.

25. Ponamus igitur verticem coni extra basin in A (Fig. 3, p. 135) incidere, ita ut sit $CA = b$ existente radio $CE = CF = c$; tum vero ex A ducantur rectae AM et AN basis tangentes ac manifestum est ex basis portione MEN, si ex singulis punctis ad A rectae ductae intelligantur, produci aream ex area circuli et trilineo $AMFN$ compositam. Deinde ex altera baseos parte MFN, si pariter ex singulis punctis ad A rectae agerentur, area prodibit itidem trilineo $AMFN$ aequalis, ita ut tota coni superficies aequalis sit areae

baseos una cum hoc trilineo bis sumto. Ad hanc igitur aream inveniendam vocemus angulum $ACM = \zeta$, et cum sit $AC = b$, erit recta tangens $AM = b \sin. \zeta$ ideoque area trianguli $ACM = \frac{1}{2} bc \sin. \zeta$, a quo auferatur area sectoris $FCM = \frac{1}{2} cc\zeta$, et remanebit area trilinei

$$AMF = \frac{1}{2} bc \sin. \zeta - \frac{1}{2} cc\zeta,$$

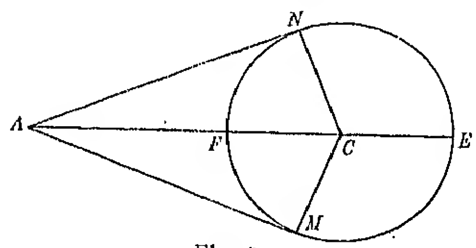

Fig. 3.

cuius duplum dabit aream trilinei $AMFN = bc \sin. \zeta - cc\zeta$, quamobrem tota superficies huius coni, cuius altitudo a est quasi infinite parva, erit

$$= \pi cc + 2bc \sin. \zeta - 2cc\zeta.$$

26. Cum igitur super hac determinatione nullum dubium superesse possit, quaeritur, cur calculus hoc casu tantopere a veritate abludat. Causa autem sine ullo dubio in formula radicali $\sqrt{(c + b \cos. \varphi)^2}$ latet; quae cum duplicem significationem involvat, alteram positivam, alteram negativam, natura nostrae quaestionis manifesto tantum valorem positivum postulat. Quare cum posuerimus $\partial S = c\partial\varphi(c + b \cos. \varphi)$, haec positio eatenus tantum valet, quatenus quantitas $c + b \cos. \varphi$ est positiva; at vero, dum angulus φ ultra rectum augetur, quia $\cos. \varphi$ fit negativus, evadere poterit $c + b \cos. \varphi = 0$, quando scilicet fit $\cos. \varphi = -\frac{c}{b}$. Quare cum supra ducta tangente AM fuerit $\cos. ACM = \cos. \zeta = \frac{c}{b}$, sequitur sumto $\varphi = \pi - \zeta$ formulam $c + b \cos. \varphi$ evanescere; sin autem angulus φ ultra hunc terminum augeatur, eius valor evadet negativus atque in locum formulae radicalis substitui debebit

$$-c - b \cos \varphi.$$

27. Ob hunc duplicem usum formulae radicalis perspicuum est integrationem formulae nostrae differentialis in duas partes distribui debere, quarum prior petenda erit ex formula

$$\partial S = c\partial\varphi(c + b \cos. \varphi),$$

cuius integrale a $\varphi = 0$ tantum usque ad terminum $\varphi = \pi - \zeta$ extendi debet;

hinc ergo colligetur

$$S = cc(\pi - \zeta) + bc \sin. \zeta;$$

alteram vero partem ex formula

$$\partial S = - c\partial\varphi (c + b \cos. \varphi)$$

deduci oportet, cuius integrale a termino $\varphi = \pi - \zeta$ usque ad terminum $\varphi = \pi$ extendi debet. Cum igitur integrale hinc oriundum sit

$$S = C - cc\varphi - bc \sin. \varphi,$$

constans ita definiatur, ut hoc integrale evanescat sumto $\varphi = \pi - \zeta$, eritque idcirco

$$C = cc(\pi - \zeta) + bc \sin. \zeta.$$

Fiat igitur nunc $\varphi = \pi$ atque altera pars nostri integralis erit

$$= bc \sin. \zeta - cc\zeta,$$

quae cum parte prius inventa praebet totam huius coni superficiem

$$\pi cc + 2bc \sin. \zeta - 2cc\zeta,$$

qui iam valor cum veritate egregie conspirat.

28. Hoc casu, quo $a = 0$, expedito facile patet etiam illis casibus, quibus altitudo a est valde parva, resolutionem bipartitam institui debere. Verum hic statim maxima se offert difficultas in evolutione formulae radicalis

$$\sqrt{aa + (c + b \cos. \varphi)^2}.$$

Cum enim altitudo a sit valde exigua, series more solito hinc nata prodit ita expressa

$$c + b \cos. \varphi + \frac{1}{2} \cdot \frac{aa}{c + b \cos. \varphi} - \frac{1 \cdot 1}{2 \cdot 4} \cdot \frac{a^4}{(c + b \cos. \varphi)^3} + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} \cdot \frac{a^6}{(c + b \cos. \varphi)^5} + \text{etc.},$$

quae series utique valde convergit, quando formula $c + b \cos. \varphi$ multum superat altitudinem a. Quoniam autem pariter transeundum est per eos

casus, quibus est $c+b\cos.\varphi=0$, post primum terminum sequentes omnes in infinitum abeunt ideoque a veritate maxime abhorrent atque adeo nullum adhuc artificium in Analysi est repertum, quo huic incommodo medela afferri posset. His igitur casibus recurrendum erit ad dimensionem practicam, qua totam superficiem coni in plures partes partiri et singularum areas seorsim exquirere solemus; id quod commodissime fiet, si superficies coni in planum explicetur, cui operationi sequens problema est destinatum.

PROBLEMA

29. *Si superficies coni scaleni in planum explicetur, indolem figurae, quae hinc nascetur, explorare.*

SOLUTIO

Concipiamus cono $AEGFH$, quem in fig. 1 et 2 sumus contemplati, chartam circumvolvi eamque iterum explicari in planum, veluti fig. 4 indicat, ubi A respondeat vertici coni, rectae autem AE et AF exhibeant latus maximum et minimum coni, ita ut area figurae EAF dimidiae superficiei conicae sit aequalis. Manentibus igitur denominationibus supra adhibitis, scilicet altitudine coni $AB=a$, obliquitate $BC=b$ et radio basis $CE=CF=c$, erit in praesenti figura latus maximum

$$AE=\sqrt{aa+(b+c)^2},$$

latus vero minimum

$$AF=\sqrt{aa+(b-c)^2},$$

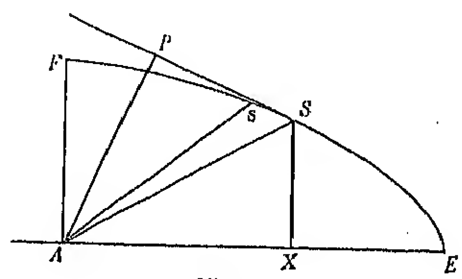

Fig. 4.

longitudo autem curvae ESF aequabitur semiperipheriae baseos coni, quae est πc. Evidens autem est istam curvam plurimum a natura circuli recedere, cuius ergo indolem et proprietates hic indagari oportet.

30. Cum triangulum elementare ASs (Fig. 2, p. 121) in ipsa superficie coni sit assumtum, id nunc in nostro plano reperietur, et quoniam rectae SP et AP in plano trianguli erant sitae, eae etiamnunc in nostrum planum

incident eritque recta SP tangens curvae in puncto S, recta vero AP erit perpendiculum ex puncto A in hanc tangentem demissum; portio vero curvae ES aequabitur arcui circulari $ES = c\varphi$, posito scilicet angulo $ECS = \varphi$. Quodsi ergo nunc has rectas vocemus $AS = v$, $AP = p$ et $SP = q$, erit ex iis, quae supra attulimus,

$$pp = aa + (c + b\cos.\varphi)^2$$

et

$$qq = bb\sin.\varphi^2 \quad \text{sive} \quad q = b\sin.\varphi,$$

unde fit

$$vv = pp + qq = aa + bb + cc + 2bc\cos.\varphi.$$

Hinc autem si vocemus aream $EAS = S$, ut ∂S exprimat aream trianguli elementaris ASs, erit, uti supra invenimus,

$$\partial S = \frac{1}{2}c\partial\varphi\sqrt{aa + (c + b\cos.\varphi)^2} = \frac{1}{2}cp\partial\varphi.$$

Quodsi iam vocemus angulum $EAS = \omega$, ut sit angulus $SAs = \partial\omega$, ob $AS = v$ area eiusdem trianguli erit $= \frac{1}{2}vv\partial\omega$, quamobrem habebitur haec aequatio $vv\partial\omega = cp\partial\varphi$ ideoque $\partial\omega = \frac{cp\partial\varphi}{vv}$ sive habebimus

$$\partial\omega = \frac{c\partial\varphi\sqrt{aa + (c + b\cos.\varphi)^2}}{aa + bb + cc + 2bc\cos.\varphi},$$

cuius ergo integrale nobis praebebit ipsum angulum EAS angulo φ respondentem; ac si tum fiat $\varphi = 180^\circ = \pi$, prodibit angulus EAF, cuius ergo determinatio maxime est difficilis, cum neque per logarithmos neque per arcus circulares expediri queat.

31. At vero haec figura continet alia symptomata, quae satis concinne exprimere licet. Primo scilicet si angulus, quem tangens SP cum recta AS constituit, vocetur $ASP = \theta$, statim habemus

$$\sin.\theta = \frac{p}{v} = \frac{\sqrt{aa + (c + b\cos.\varphi)^2}}{\sqrt{(aa + bb + cc + 2bc\cos.\varphi)}}$$

et

$$\cos.\theta = \frac{q}{v} = \frac{b\sin.\varphi}{\sqrt{(aa + bb + cc + 2bc\cos.\varphi)}},$$

unde patet in ipso puncto E, ubi $\varphi=0$, fieri cos. $\theta=0$ ideoque rectam AE ad curvam in E esse normalem, quod idem quoque evenit in puncto F, ubi $\varphi=\pi$, ita ut in ambobus terminis E et F rectae AE et AF curvae normaliter insistant; in punctis autem intermediis rectae AS cum curva angulos obliquos constituent, quemadmodum ex quantitate tangentis SP est manifestum. Ubi imprimis notasse iuvabit, si punctum S capiatur in ipso puncto G (Fig. 1), ubi est $\varphi=90°$, tum quantitatem tangentis $SP=q$ fore $=b$ ideoque ipsi obliquitati coni aequalem. In omnibus autem reliquis punctis ista tangens $SP=q$ minor erit quam obliquitas b.

32. Praeterea vero etiam ipsam curvaturam nostrae curvae ESF in singulis punctis S satis concinne exprimere licet. Si enim radium osculi in puncto S designemus littera r, constat eum ex perpendiculo in tangentem $AP=p$ ita exprimi, ut sit $r=\frac{v\partial v}{\partial p}$. Cum igitur sit

$$v\partial v=-bc\partial\varphi\sin.\varphi$$

et

$$p\partial p=-b\partial\varphi\sin.\varphi(c+b\cos.\varphi)$$

ideoque

$$\partial p=-\frac{b\partial\varphi\sin.\varphi(c+b\cos.\varphi)}{p},$$

his valoribus substitutis reperitur radius osculi

$$r=\frac{cp}{c+b\cos.\varphi},$$

unde sequitur in ipso puncto E, ubi $\varphi=0$, radium osculi fore

$$r=\frac{cp}{c+b}=\frac{c\sqrt{aa+(c+b)^2}}{c+b};$$

at vero in altero termino F, ubi $\varphi=\pi$, radius osculi erit

$$r=\frac{cp}{c-b}=\frac{c\sqrt{aa+(c-b)^2}}{c-b}.$$

Unde patet, si fuerit $b>c$, hoc est iis casibus, quibus altitudo AB extra basin cadit, tum radium osculi in F fore negativum ideoque curvam in hoc

loco convexitatem versus A obvertere; contra autem, quamdiu fuerit $b < c$, tum totam curvam ubique versus A fore concavam.

33. Quodsi porro longitudinem curvae ES ponamus $= s$, ita ut sit $s = c\varphi$, notum est formulam integralem $\int \frac{\partial s}{r}$ exprimere amplitudinem arcus curvae ES; quae si designetur littera ψ, erit $\partial\psi = \frac{\partial s}{r}$, quamobrem substitutis valoribus pro ∂s et r inventis habebimus

$$\partial\psi = \frac{c\partial\varphi(c + b\cos\varphi)}{\sqrt{aa + (c + b\cos\varphi)^2}},$$

cuius formulae integratio, etiamsi pariter expediri nequeat, tamen multo simplicior est censenda illa, qua $\partial\omega$ exprimebatur. Invento autem hoc angulo ψ ex eo quoque ipsum illum angulum ω definire licebit. Ducta enim ex S ad rectam AE perpendiculari SX angulus ESX ipsam curvae amplitudinem metitur; quare, cum etiam angulus $ASP = \theta$ sit cognitus, erit angulus $ASX = 180^\circ - \theta - \psi$; qui cum etiam sit $= 90^\circ - \omega$, reperietur ipse angulus

$$\omega = \theta + \psi - 90^\circ$$

sicque integratione formulae illius difficillimae pro $\partial\omega$ inventae supersedere poterimus.

34. Ex his iam, quae hactenus sunt allata, ipsa curva ESF haud difficulter in plano describi poterit; quae si in plures partes dividatur, singularum partium areae facili negotio practice mensurari poterunt, quae in unam summam collectae dabunt superficiem coni scaleni propositi. Caeterum hic silentio non est praetereundum, quoniam haec figura per explanationem chartae tam facile exhiberi potest, hinc eximium exemplum curvae maxime transcendentis obtineri, cuius nihilominus descriptio facillime expediri queat.

ADDITAMENTUM AD § 21

Quodsi formulas in § 21 traditas evolvamus atque simili modo, ut ibi coepimus, summas terminorum quintorum, sextorum et sequentium actu definiamus, seriem haud inelegantem pro superficie coni scaleni exhibere

poterimus. Quodsi enim brevitatis gratia ponamus

$$c = ax \quad \text{et} \quad \sqrt{(1 + xx)} = u,$$

tota coni scaleni superficies erit

$$= \pi aaxuV$$

denotante V summam sequentis seriei

$$\begin{aligned} V = 1 &+ \frac{1}{2^2} \cdot \frac{bb}{1 \cdot aa} \cdot \frac{1}{u^4} - \frac{1 \cdot 3}{2^2 \cdot 4^2} \cdot \frac{b^4}{3 \cdot a^4} \left(\frac{1 \cdot 3}{u^6} - \frac{3 \cdot 5xx}{u^8} \right) \\ &+ \frac{1 \cdot 3 \cdot 5}{2^2 \cdot 4^2 \cdot 6^2} \cdot \frac{b^6}{5 \cdot a^6} \left(\frac{1 \cdot 3 \cdot 5}{u^8} - 2 \cdot \frac{3 \cdot 5 \cdot 7xx}{u^{10}} + \frac{5 \cdot 7 \cdot 9x^4}{u^{12}} \right) \\ &+ \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} \cdot \frac{b^8}{7 \cdot a^8} \left(\frac{1 \cdot 3 \cdot 5 \cdot 7}{u^{10}} - 3 \cdot \frac{3 \cdot 5 \cdot 7 \cdot 9xx}{u^{12}} + 3 \cdot \frac{5 \cdot 7 \cdot 9 \cdot 11x^4}{u^{14}} - \frac{7 \cdot 9 \cdot 11 \cdot 13x^6}{u^{16}} \right) \\ &+ \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10^2} \cdot \frac{b^{10}}{9 \cdot a^{10}} \left(\begin{array}{l} \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{u^{12}} - 4 \cdot \frac{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11xx}{u^{14}} + 6 \cdot \frac{5 \cdot 7 \cdot 9 \cdot 11 \cdot 13x^4}{u^{16}} \\ \quad - 4 \cdot \frac{7 \cdot 9 \cdot 11 \cdot 13 \cdot 15x^6}{u^{18}} + \frac{9 \cdot 11 \cdot 13 \cdot 15 \cdot 17x^8}{u^{20}} \end{array} \right) + \text{etc.} \end{aligned}$$

Evidens autem est hanc seriem iis tantum casibus usum praestare, quibus quantitas bb multo minor est quam formula $aa + cc$; quando autem propemodum est aequalis vel adeo maior, tum necessario confugiendum erit ad descriptionem illam practicam, quam supra exposuimus.

DE BINIS CURVIS ALGEBRAICIS INVENIENDIS QUARUM ARCUS INDEFINITE INTER SE SINT AEQUALES

Convent. exhib. die 20 Iunii 1776

Commentatio 693 indicis ENESTROEMIANI

Nova acta academiae scientiarum Petropolitanae 4 (1786), 1789, p. 96–103

Summarium ibidem p. 116–117

SUMMARIUM

En désignant par X et x les abscisses, et par Y et y les ordonnées de deux courbes algébriques, le problème géométrique que feu M. EULER traite ici dans ce mémoire, se réduit à cette question purement analytique: Quelles sont les quatre fonctions algébriques d'une nouvelle variable z, qu'il faut prendre pour X, Y, x, y, afin que

$$\partial X^2 + \partial Y^2 = \partial x^2 + \partial y^2.$$

En mettant

$$X = p + q, \qquad x = p - q,$$
$$Y = r - s, \qquad y = r + s,$$

cette condition se réduit à la suivante: $\partial p \partial q = \partial r \partial s$, qu'on peut remplir, comme l'Auteur observe, d'une infinité de manières, si l'on veut se contenter de solutions particulières.

Quant à la solution générale du problème, M. EULER en donne deux. La première se réduit aux règles suivantes: 1) A la place de q et r prenez deux fonctions quelconques algébriques de z et faites $u = \frac{\partial q}{\partial r}$. 2) Prenez pour $\int p \partial u = v$ aussi une fonction quelconque algébrique de z et 3) donnez à p et s les valeurs suivantes: $p = \frac{\partial v}{\partial u}$, $s = \frac{u \partial v}{\partial u} - v$ et les coordonnées des deux courbes cherchées seront

$$X = \frac{\partial v}{\partial u} + q, \qquad x = \frac{\partial v}{\partial u} - q,$$
$$Y = r - \frac{u \partial v}{\partial u} + v, \qquad y = r + \frac{u \partial v}{\partial u} - v.$$

De cette maniere, ajoute l'Auteur, si les trois fonctions arbitraires q, r et v pouvoient être prises de manière que l'une des deux courbes fût une courbe déterminée, par exemple l'ellipse, ou l'hyperbole, il seroit dans notre pouvoir de trouver, moyennant cette solution, une autre courbe dont un arc fût égal en longueur à un arc de l'une ou de l'autre des deux courbes mentionnées données; mais il doute que l'Analyse puisse jamais atteindre ce degré de perfection.

La seconde solution, pour être générale, suppose possible la résolution des équations algébriques de tout ordre. Car l'Auteur observe que mettant $\frac{\partial s}{\partial p} = \frac{\partial q}{\partial r} = z$ quelque équation algébrique qu'on adopte entre p et s, de même qu'entre q et r, on aura p et s aussi bien que q et r exprimées par des fonctions de la même variable z. Cette solution cède donc le rang à la premiere. Cependant elle n'est pas sans utilité, puisqu'elle fournit un moyen très-simple et très-élégant de construire les deux courbes dont deux arcs pris indéfiniment sont de la même longueur.

1. Sint AY (Fig. 1) et ay (Fig. 2) huiusmodi binae curvae algebraicae, quarum arcus AY et ay inter se sint aequales, ac pro priore vocentur

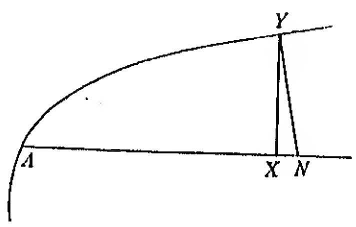

Fig. 1.

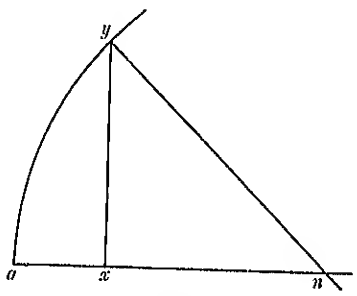

Fig. 2.

coordinatae orthogonales $AX = X$ et $XY = Y$, pro altera vero $ax = x$ et $xy = y$; tum igitur requiritur, ut sit

$$\sqrt{(\partial X^2 + \partial Y^2)} = \sqrt{(\partial x^2 + \partial y^2)}.$$

Hunc in finem introducta in calculum nova variabili z quaestio huc redit, cuiusmodi quatuor functiones algebraicae istius quantitatis z pro quaternis illis coordinatis X, Y et x, y accipi debeant, ut utrinque elementa curvae inter se evadant aequalia sive ut fiat $\partial X^2 + \partial Y^2 = \partial x^2 + \partial y^2$. Talibus enim functionibus inventis manifestum est inde pro utraque curva aequationes algebraicas inter binas coordinatas erui posse, ita ut ambae curvae prodituraesint algebraicae.

2. Quo nunc hanc conditionem facilius adimpleamu
dinatas sequenti modo per partes exprimamus

$$X = p + q, \qquad x = p - q,$$
$$Y = r - s, \qquad y = r + s;$$

sic enim pro priori curva reperietur

$$\partial X^2 + \partial Y^2 = \partial p^2 + \partial q^2 + \partial r^2 + \partial s^2 + 2\partial p\partial q$$

pro altera vero curva erit

$$\partial x^2 + \partial y^2 = \partial q^2 + \partial p^2 + \partial r^2 + \partial s^2 - 2\partial q\partial p +$$

quae formulae cum inter se debeant esse aequales, sati
aequationi $\partial p\partial q = \partial r\partial s$. Sicque tota quaestio perducta
functiones algebraicas ipsius z pro litteris p, q, r, s inve

$$\partial p\partial q = \partial r\partial s,$$

cui conditioni haud difficulter infinitis modis satisfieri
tionibus particularibus acquiescere vellemus.

3. Veluti si ponamus

$$p = Az^\alpha, \quad q = Bz^\beta, \quad r = Cz^\gamma, \quad s = Dz^\delta,$$

effici debet, ut fiat

$$AB\alpha\beta z^{\alpha+\beta-2}\partial z^2 = CD\gamma\delta z^{\gamma+\delta-2}\partial z^2.$$

Hic igitur duabus conditionibus erit satisfaciendum: prim
$\alpha + \beta = \gamma + \delta$, tum vero $AB\alpha\beta = CD\gamma\delta$. Ut nunc pri
modissime satisfaciamus, statuamus

$$\alpha = \lambda + \mu, \quad \beta = \lambda - \mu, \quad \gamma = \lambda + \nu, \quad \delta =$$

sic enim fiet $\alpha + \beta = \gamma + \delta = 2\lambda$. His autem valoribus
conditio postulat, ut fiat

$$AB(\lambda^2 - \mu^2) = CD(\lambda^2 - \nu^2)$$

sive $\frac{AB}{CD} = \frac{\lambda^2 - \nu^2}{\lambda^2 - \mu^2}$; cui conditioni nitidissime et generalissime satisfaciemus ponendo

$$A = fg(\lambda + \nu), \qquad C = fh(\lambda + \mu),$$
$$B = hk(\lambda - \nu), \qquad D = gk(\lambda - \mu),$$

ubi tam tres numeros λ, μ et ν quam quatuor quantitates f, g, h et k prorsus pro lubitu assumere licet; quamobrem hinc pro priore curva nanciscemur coordinatas

$$X = fg(\lambda + \nu)z^{\lambda+\mu} + hk(\lambda - \nu)z^{\lambda-\mu},$$
$$Y = fh(\lambda + \mu)z^{\lambda+\nu} - gk(\lambda - \mu)z^{\lambda-\nu},$$

pro altera vero curva habebimus

$$x = fg(\lambda + \nu)z^{\lambda+\mu} - hk(\lambda - \nu)z^{\lambda-\mu},$$
$$y = fh(\lambda + \mu)z^{\lambda+\nu} + gk(\lambda - \mu)z^{\lambda-\nu}.$$

Pro utraque autem curva erit elementum curvae

$$= \sqrt{\partial p^2 + \partial q^2 + \partial r^2 + \partial s^2}.$$

4. Verum hic nobis imprimis est propositum in solutionem generalem inquirere, quae omnes plano speciales in se complectatur, ideoque conditioni inventae $\partial r \partial s = \partial p \partial q$ generalissime erit satisfaciendum. Cum igitur hinc sit $\partial s = \frac{\partial p \partial q}{\partial r}$, eiusmodi functiones pro p, q, r investigari oportet, ut ista formula differentialis $\frac{\partial p \partial q}{\partial r}$ integrationem admittat; ubi quidem, quoniam haec unica conditio est adimplenda, facile intelligitur ex ternis quantitatibus p, q et r binas arbitrio nostro penitus relinqui. Quamobrem assumtis pro q et r functionibus quibuscunque algebraicis ipsius z inde colligatur valor formulae $\frac{\partial q}{\partial r}$, qui ergo itidem erit functio algebraica ipsius z simulque cognita, quam indicemus littera u, ita ut sit $\frac{\partial q}{\partial r} = u$ atque adeo $\partial s = u \partial p$, cui igitur conditioni satisfieri oportet.

5. Cum igitur hic u tanquam functio cognita ipsius z spectetur, totum negotium redit ad functionem p investigandam. Quare, cum sit $s = \int u \partial p$, per reductionem notissimam habebimus

$$s = pu - \int p \partial u,$$

ita ut formula differentialis $p\partial u$ integrabilis reddi debeat. Statuatur ergo $\int p\partial u = v$ existente v functione pariter algebraica ipsius z, unde ergo fiet $p = \frac{\partial v}{\partial u}$ hincque porro

$$s = pu - v = \frac{u\partial v}{\partial u} - v,$$

sicque universe conditioni praescriptae erit satisfactum atque adeo pro r functionem quamcunque algebraicam ipsius z pro arbitrio assumere licebit.

6. Ecce ergo quaestionis nostrae propositae solutio generalissima sequenti modo adornari poterit. 1) Pro litteris q et r sumantur pro lubitu functiones quaecunque algebraicae ipsius z, ex quibus deducatur quantitas

$$u = \frac{\partial q}{\partial r}.$$

2) Accipiatur etiam pro v functio quaecunque algebraica ipsius z, ita ut adeo tres functiones ipsius z arbitrio nostro penitus permittantur. 3) His igitur constitutis binae reliquae litterae p et s ita accipiantur, ut sit

$$p = \frac{\partial v}{\partial u} \quad \text{et} \quad s = \frac{u\partial v}{\partial u} - v.$$

Quibus inventis ambae curvae quaesitae ita determinabuntur, ut earum coordinatae orthogonales futurae sint:

Pro curva AY	Pro curva ay
$X = \frac{\partial v}{\partial u} + q$	$x = \frac{\partial v}{\partial u} - q$
$Y = r - \frac{u\partial v}{\partial u} + v$	$y = r + \frac{u\partial v}{\partial u} - v$

haecque ergo solutio ita est generalis, ut omnes plane casus possibiles in se complectatur.

7. Cum igitur sit utriusque curvae elementi quadratum

$$= \partial p^2 + \partial q^2 + \partial r^2 + \partial s^2,$$

quoniam primo est $\partial q = u\partial r$, tum vero $s = pu - v$, ob $\partial v = p\partial u$ erit

$\partial s = u \partial p$; unde his valoribus substitutis obtinebitur elementum utriusque curvae

$$= \sqrt{(1 + uu)(\partial p^2 + \partial r^2)}.$$

8. Cum igitur porro sit $\partial q = u \partial r$ et $\partial s = u \partial p$, erit pro priore curva AY formula

$$\frac{\partial X}{\partial Y} = \frac{\partial p + u \partial r}{\partial r - u \partial p},$$

quae ducta in curvae AY normali YN exprimit tangentem anguli ANY. Simili modo in altera ay, si pariter ducatur normalis yn, erit anguli any tangens

$$= \frac{\partial x}{\partial y} = \frac{\partial p - u \partial r}{\partial r + u \partial p}.$$

Quamobrem si introducamus binos angulos φ et θ, ita ut sit

$$\operatorname{tang.} \varphi = \frac{\partial p}{\partial r} \quad \text{et} \quad \operatorname{tang.} \theta = u,$$

evadet

$$\frac{\partial X}{\partial Y} = \frac{\operatorname{tang.} \varphi + \operatorname{tang.} \theta}{1 - \operatorname{tang.} \varphi \operatorname{tang.} \theta} = \operatorname{tang.} (\varphi + \theta)$$

et

$$\frac{\partial x}{\partial y} = \frac{\operatorname{tang.} \varphi - \operatorname{tang.} \theta}{1 + \operatorname{tang.} \varphi \operatorname{tang.} \theta} = \operatorname{tang.} (\varphi - \theta).$$

Unde manifestum est angulos ANY et any, quibus utriusque curvae amplitudo mensuratur, fore $ANY = \varphi + \theta$ et $any = \varphi - \theta$.

9. Hinc igitur intelligitur ambas nostras curvas communi amplitudine gaudere non posse, nisi fuerit angulus $\theta = 0$; tum autem foret $u = 0$ ideoque ob $\partial q = u \partial r$ et $\partial s = u \partial p$ ambae quantitates q et s forent constantes, quae ergo ponantur $q = a$ et $s = b$, unde propterea prodiret $X = p + a$ et $Y = r - b$, tum vero $x = p - a$ et $y = r + b$; sicque foret $x = X - 2a$ et $y = Y + 2b$, unde manifestum est ambas curvas prorsus fore easdem, verum coordinatas tantum ad alios axes referri.

10. Cum igitur hoc problema felicissimo cum successu generaliter expediverimus, si ternae functiones arbitrariae q, r et v ita definiri possent, ut altera curva oriatur data, veluti sive ellipsis sive hyperbola, tum simul inveniretur alia curva propositae aequalis. Verum talem methodum vix ac ne vix quidem sperare licet, quandoquidem problema generale curvam quamcunque datam in alias diversas eiusdem longitudinis transformandi vires Analyseos superare videtur.

ALIA SOLUTIO QUAESTIONIS PROPOSITAE

11. Cum tota solutio perducta sit ad hanc aequationem

$$\partial p\,\partial q = \partial r\,\partial s \quad \text{sive} \quad \frac{\partial s}{\partial p} = \frac{\partial q}{\partial r},$$

statuatur tam $\frac{\partial s}{\partial p} = z$ quam $\frac{\partial q}{\partial r} = z$ et, quaecunque accipiatur aequatio algebraica inter p et s, qua littera s definiatur per certam functionem ipsius p, erit etiam $\frac{\partial s}{\partial p}$ certa functio ipsius p, qua ipsi z aequali posita quantitas p ideoque et altera s per z determinabitur. Simili modo sumta inter q et r aequatione algebraica quacunque, ex qua q definiatur per certam functionem ipsius r, fiet etiam $\frac{\partial q}{\partial r}$ certa functio ipsius r, quae posita $= z$ dabit itidem tam r quam q per functiones ipsius z expressas. Inventis autem his quatuor functionibus p, q, r, s ambae curvae quaesitae ita determinabuntur per suas utraque coordinatas, ut sit

$$X = p + q, \qquad Y = r - s,$$
$$x = p - q, \qquad y = r + s.$$

12. Quoniam vero ista solutio postulat resolutionem aequationum omnis generis, siquidem generalis esse debeat, prior solutio huic sine dubio longe est anteferenda. Interim tamen etiam haec solutio usu non caret, dum nobis egregiam constructionem geometricam binarum curvarum, quae quaeruntur, suppeditat, quae ita se habet.

CONSTRUCTIO GEOMETRICA CURVARUM QUAESITARUM

13. Super communi axe describantur binae curvae algebraicae quaecunque bs et cq (Fig. 3), in quibus perpetuo capiantur bina puncta s et q, ubi

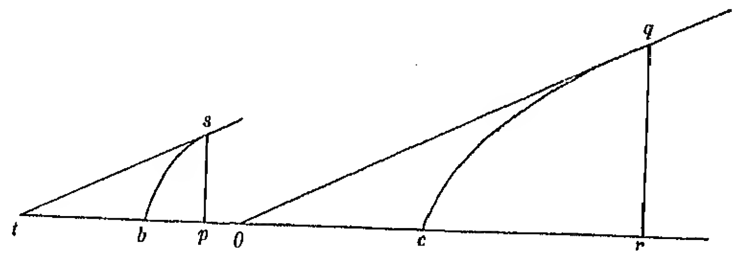

Fig. 3.

tangentes st et $q0$ inter se fiant parallelae; tum ductis applicatis sp et qr habebuntur quatuor nostrae quantitates, scilicet

$$bp = p, \quad ps = s \quad \text{et} \quad cr = r, \quad rq = q.$$

Quia enim $\frac{\partial s}{\partial p}$ exprimit tangentem anguli ad t et $\frac{\partial q}{\partial r}$ tangentem anguli ad 0, cum hi anguli sint aequales, erit utique

$$\frac{\partial s}{\partial p} = \frac{\partial q}{\partial r} \quad \text{ideoque} \quad \partial p \partial q = \partial r \partial s,$$

uti requiritur. Quamobrem ex his duabus curvis pro arbitrio assumtis binae curvae quaesitae hoc modo construentur:

Pro curva AY	Pro curva ay
$AX = bp + rq$	$ax = bp - rq$
$XY = cr - ps$	$xy = cr + ps$

quae constructio ob elegantiam utique notatu maxime est digna.

14. Pro curva bs sumamus parabolam hac aequatione contentam $ss = 2ap$, pro altera autem cq circulum aequatione $qq = 2ar - rr$ expressum; tum igitur erit

$$\frac{\partial s}{\partial p} = \frac{a}{\sqrt{2ap}} \quad \text{et} \quad \frac{\partial q}{\partial r} = \frac{a - r}{\sqrt{2ar - rr}},$$

quae duae quantitates inter se aequales esse debent; perinde enim est, sive sibi immediate aequales statuantur sive utraque ipsi z aequalis statuatur. Hoc modo omnia ad solam quantitatem r revocare licebit, quandoquidem habebimus

$$p = \frac{a(2ar - rr)}{2(a-r)^2}, \quad q = \sqrt{(2ar - rr)} \quad \text{et} \quad s = \frac{a\sqrt{(2ar - rr)}}{a - r}.$$

Unde pro binis curvis has nanciscimur coordinatas (Fig. 1 et 2, p. 143):

Pro curva AY

$$X = \frac{a(2ar - rr)}{2(a-r)^2} + \sqrt{(2ar - rr)}$$

$$Y = r - \frac{a\sqrt{(2ar - rr)}}{a - r}$$

Pro curva ay

$$x = \frac{a(2ar - rr)}{2(a-r)^2} - \sqrt{(2ar - rr)}$$

$$y = r + \frac{a\sqrt{(2ar - rr)}}{a - r}.$$

Ubi quantitatem r tantum usque ad $r = a$ augere licet, quia tum ambae curvae in infinitum excurrent.

DE INNUMERIS CURVIS ALGEBRAICIS QUARUM LONGITUDINEM PER ARCUS PARABOLICOS METIRI LICET

Convent. exhib. die 3 Iunii 1776

Commentatio 638 indicis ENESTROEMIANI

Nova acta academiae scientiarum Petropolitanae 5 (1787), 1789, p. 59—70

Summarium ibidem p. 65—67

SUMMARIUM

Dans un Mémoire intitulé: *Theoremata quaedam analytica, quorum demonstratio adhuc desideratur*, qu'on trouve dans les Opuscules analytiques, Tom. II. pag. 76[1]), feu M. EULER avoit entre autres avancé les deux propositions suivantes:

1. Qu'il n'y ait point de courbe algébrique dont la longueur pût être exprimée simplement par des logarithmes.

2. Qu'à l'exception du cercle il n'y ait point de courbe algébrique dont chaque arc pût être mesuré par un arc de cercle.

Il avoit même tâché de mettre la vérité de ces deux propositions hors de doute, par des raisonnemens aussi concluans que profonds, sans pouvoir cependant donner à ses démonstrations toute la rigueur requise.

La chose se réduit, comme chacun voit aisément, à trouver, pour les coordonnées x et y d'une courbe, de telles fonctions de v, qu'il y ait $\sqrt{(\partial x^2+\partial y^2)} = V\partial v$, et à montrer dans quels cas le problème n'a point de solution, comme cela arrive lorsque $V\partial v = \frac{\partial v}{v}$; ou dans quels cas il n'admet qu'une seule solution, comme cela a lieu lorsque $V\partial v = \frac{\partial v}{\sqrt{(1-vv)}}$; ou enfin, dans quels cas le nombre des solutions est infini, comme il l'est en mettant

1) Vide p. 78. A. K.

[illegible] $v\sqrt{(1+vv)}$, $\int \partial v\sqrt{(1+vv)}$ exprimant un arc parabolique dont les coordonnées sont v et $\frac{1}{2}vv$. L'Auteur, qui a déjà renoncé à la démonstration des deux premiers cas, s'attache ici à traiter le troisième, en résolvant le problème suivant:

Trouver une infinité de courbes algébriques dont les arcs puissent être exprimés par des arcs paraboliques.

Il donne trois solutions différentes de ce problème dont chacune fournit une infinité de courbes algébriques satisfaisantes. Nous tâcherons de donner aux lecteurs de ces Extraits une idée de la troisième, comme de la plus simple.

Comme $\sqrt{(\partial x^2+\partial y^2)}=\partial v\sqrt{(1+vv)}$, on mettra $v=\sin.\theta$, pour avoir

$$\sqrt{(\partial x^2+\partial y^2)}=\partial\theta\cos.\theta\sqrt{(\cos.\theta^2+2\sin.\theta^2)}.$$

Or toutes les fois que $\int V\partial v$ pourra être réduit à la forme $\int\partial v\sqrt{(P^2+Q^2)}$, on aura

$$\partial x=\partial v(P\sin.\varphi+Q\cos.\varphi)$$
$$\partial y=\partial v(P\cos.\varphi-Q\sin.\varphi),$$

et ces deux formules doivent être intégrables, ce qu'on effectuera en donnant à φ des valeurs propres à cet effet. Dans le cas présent, où $P=\cos.\theta$ et $Q=\sin.\theta\cdot\sqrt{2}=n\sin.\theta$, on aura

$$\frac{\partial x}{\partial\theta\cos.\theta}=\cos.\theta\sin.\varphi+n\sin.\theta\cos.\varphi,$$
$$\frac{\partial y}{\partial\theta\cos.\theta}=\cos.\theta\cos.\varphi-n\sin.\theta\sin.\varphi;$$

expressions qui, par quelques réductions assez connues, se laissent transformer ainsi

$$\frac{4\partial x}{\partial\theta}=2\sin.\varphi+(n+1)\sin.(\varphi+2\theta)-(n-1)\sin.(\varphi-2\theta),$$
$$\frac{4\partial y}{\partial\theta}=2\cos.\varphi+(n+1)\cos.(\varphi+2\theta)-(n-1)\cos.(\varphi-2\theta).$$

Ces expressions deviennent intégrables, en mettant $\varphi=\alpha+\lambda\theta$, de sorte qu'on aura sans peine les deux coordonnées d'une infinité de courbes algébriques dont les arcs pourront être exprimés par des arcs paraboliques; car α et λ peuvent recevoir une infinité de valeurs différentes.

1. Non ita pridem[1]) ausus sum duo theoremata prorsus memorabilia in medium proferre, quorum altero statui *nullam dari curvam algebraicam, cuius longitudo indefinita per quempiam logarithmum exprimi queat*, altero vero affir-

1) L. Euleri Commentatio 590 (indicis Enestroemiani); vide p. 78. A. K.

mavi *praeter circulum nullas alias dari curvas algebraicas, quarum longitudo cuipiam arcui circulari esset aequalis.* Veritatem equidem horum theorematum gravissimis rationibus confirmare sum annisus; interim tamen fateri cogor omnes has rationes a solida demonstratione, cuiusmodi in geometria desiderari solet, adhuc plurimum abesse.

2. Facile autem intelligitur totum hoc negotium felicissimo successu confectum iri, si sequens problema resolvere liceret:

Proposita formula differentiali quacunque $V dv$, ubi V sit functio quaecunque data algebraica ipsius v, invenire pro binis coordinatis x et y eiusmodi functiones algebraicas ipsius v, ut inde evadat $\sqrt{(\partial x^2 + \partial y^2)} = V\partial v$.

Tum enim integrale $\int V\partial v$ utique exprimeret longitudinem curvae cuiusdam algebraicae. Hic scilicet res eo rediret, ut ostenderetur, quibusnam casibus hoc problema vel nullam plane solutionem admitteret, quemadmodum evenire statuo casu $V\partial v = \frac{\partial v}{v}$, vel unicam tantum solutionem, veluti casu $V\partial v = \frac{\partial v}{\sqrt{(1-vv)}}$ sive etiam $V\partial v = \frac{\partial v}{1+vv}$, vel denique, quibusnam casibus hoc problema innumerabiles solutiones recipere posset, quemadmodum ostensurus sum pro casu $V\partial v = \partial v\sqrt{(1+vv)}$, quandoquidem eius integrale $\int \partial v\sqrt{(1+vv)}$ exprimit arcum parabolicum, cuius quippe coordinatae sunt v et $\frac{1}{2}vv$.

3. Ante autem quam hoc problema particulare suscipiam, duplicem methodum aperiam, qua problema generale tractari conveniat. Ac primo quidem proposita aequatione

$$\sqrt{(\partial x^2 + \partial y^2)} = V\partial v$$

dispiciatur, num forte eiusmodi functionem ipsius v, quae sit U, explorare liceat, ut hae duae formulae

$$\partial x = \frac{V\partial v\sqrt{(A+U)}}{\sqrt{(A+B)}} \quad \text{et} \quad \partial y = \frac{V\partial v\sqrt{(B-U)}}{\sqrt{(A+B)}}$$

fiant integrabiles; quoniam enim inde fit $\partial x^2 + \partial y^2 = V^2\partial v^2$, quaestioni foret satisfactum. Vel etiam quaeratur eiusmodi angulus φ, qui rationem algebraicam teneat ad variabilem v, ita ut ambae istae formulae $V\partial v \sin.\varphi$ et $V\partial v \cos.\varphi$ evadant integrabiles, quoniam hinc fieret

$$x = \int V\partial v \sin.\varphi \quad \text{et} \quad y = \int V\partial v \cos.\varphi.$$

4. Quando autem hoc tentamen nullo modo succedit, dispiciatur, utrum formula proposita $V\partial v$ ad huiusmodi formam reduci queat $\partial v \sqrt{(P^2+Q^2)}$; tum enim statim haberetur solutio $x=\int P\partial v$ et $y=\int Q\partial v$, si modo hae formulae essent integrabiles. At vero multo generalius solutionem tentare licebit statuendo

$$\frac{\partial x}{\partial v}=\frac{P\sqrt{(A+U)}-Q\sqrt{(B-U)}}{\sqrt{(A+B)}},$$
$$\frac{\partial y}{\partial v}=\frac{P\sqrt{(B-U)}+Q\sqrt{(A+U)}}{\sqrt{(A+B)}},$$

ubi totum negotium eo redit, ut pro U eiusmodi functio ipsius v investigetur, qua istae duae formulae integrabiles reddantur. Vel etiam simplicius res redigi poterit ad inventionem cuiuspiam anguli φ, ut istae ambae formulae integrationem admittant

$$\partial x=\partial v(P\sin.\varphi+Q\cos.\varphi),$$
$$\partial y=\partial v(P\cos.\varphi-Q\sin.\varphi),$$

siquidem hinc prodibit

$$\partial x^2+\partial y^2=\partial v^2(P^2+Q^2).$$

Verum fatendum est has regulas ita esse comparatas, ut, si eas ad formulas determinatas applicare velimus, aqua nobis plerumque haereat.

5. His igitur praemissis problema, cuius solutionem pollicemur, aggrediamur.

PROBLEMA

Invenire innumerabiles curvas algebraicas, quarum longitudinem per arcus parabolicos exprimere liceat, sive ut positis binis coordinatis x et y fiat

$$\sqrt{(\partial x^2+\partial y^2)}=\partial v\sqrt{(1+vv)}$$

simulque ipsae coordinatae x et y prodeant functiones algebraicae ipsius v.

SOLUTIO

6. Quodsi hanc formulam cum generali ante allegata comparemus, erit $P=1$ et $Q=v$, unde statim colligitur $\partial x=\partial v$ et $\partial y=v\partial v$ hincque porro $x=v$ et $y=\frac{1}{2}vv$, ergo $y=\frac{1}{2}xx$, quae est aequatio pro ipsa parabola.

Problema autem nostrum postulat, ut innumeras alias eiusmodi curvas investigemus, quarum longitudo pari formula exprimatur; sequamur igitur formulam priorem § 4 traditam, unde pro praesenti casu erit

$$\frac{\partial x}{\partial v} = \frac{\sqrt{(A+U)} - v\sqrt{(B-U)}}{\sqrt{(A+B)}} \quad \text{et} \quad \frac{\partial y}{\partial v} = \frac{\sqrt{(B-U)} + v\sqrt{(A+U)}}{\sqrt{(A+B)}},$$

quae ambae formulae infinitis modis integrabiles reddi possunt, primo scilicet, si statuamus $U = v$, deinde vero etiam, si fuerit $U = \sqrt{v}$, porro quoque simili modo, si sumatur $U = \sqrt[3]{v}$ vel $U = \sqrt[4]{v}$ vel in genere $U = \sqrt[i]{v}$, si modo exponens i fuerit integer positivus.

EVOLUTIO CASUS QUO $U = v$

7. Hic igitur totum negotium redit ad integrationem talium duarum formularum

$$\int \partial v \sqrt{(\alpha + \beta v)} \quad \text{et} \quad \int v \partial v \sqrt{(\alpha + \beta v)}.$$

Statuamus igitur

$$\sqrt{(\alpha + \beta v)} = t$$

eritque

$$v = \frac{tt - \alpha}{\beta} \quad \text{et} \quad \partial v = \frac{2t\partial t}{\beta};$$

hinc ergo pro formula priore fiet

$$\partial v \sqrt{(\alpha + \beta v)} = \frac{2tt\partial t}{\beta},$$

pro altera vero formula erit

$$v \partial v \sqrt{(\alpha + \beta v)} = \frac{2tt\partial t}{\beta\beta}(tt - \alpha),$$

quocirca integrando eliciemus

$$\text{I.} \quad \int \partial v \sqrt{(\alpha + \beta v)} = \frac{2t^3}{3\beta} = \frac{2}{3\beta}(\alpha + \beta v)^{\frac{3}{2}}$$

et

$$\text{II.} \quad \int v \partial v \sqrt{(\alpha + \beta v)} = \frac{2t^5}{5\beta\beta} - \frac{2\alpha t^3}{3\beta\beta} = \frac{2t^3}{15\beta\beta}(3tt - 5\alpha)$$

sive

$$\int v \partial v \sqrt{(\alpha + \beta v)} = \frac{2(\alpha + \beta v)^{\frac{3}{2}}}{15\beta\beta}(3\beta v - 2\alpha).$$

8. Nunc igitur tantum superest, ut formulae supra exhibitae iuxta has regulas expediantur, quae ita se habebunt

$$1.\ \int \partial v \sqrt{(A+v)} = \frac{2}{3}(A+v)^{\frac{3}{2}},$$

$$2.\ \int \partial v \sqrt{(B-v)} = -\frac{2}{3}(B-v)^{\frac{3}{2}},$$

$$3.\ \int v \partial v \sqrt{(A+v)} = \frac{2}{15}(A+v)^{\frac{3}{2}}(3v-2A),$$

$$4.\ \int v \partial v \sqrt{(B-v)} = -\frac{2}{15}(B-v)^{\frac{3}{2}}(3v+2B).$$

His igitur valoribus substitutis ambae coordinatae x et y ita reperientur expressae

$$x\sqrt{(A+B)} = +\frac{2}{3}(A+v)^{\frac{3}{2}} + \frac{2}{15}(B-v)^{\frac{3}{2}}(3v+2B)$$

et

$$y\sqrt{(A+B)} = -\frac{2}{3}(B-v)^{\frac{3}{2}} + \frac{2}{15}(A+v)^{\frac{3}{2}}(3v-2A).$$

9. Hac igitur ratione ambas coordinatas x et y per communem variabilem v algebraice expressas sumus consecuti, id quod ad curvam construendam sufficit, quandoquidem pro quolibet valore ipsius v quantitates utriusque coordinatae assignare licet. Sin autem quantitatem v eliminare vellemus, in calculos molestissimos illaberemur, vix adeo extricabiles, atque aequatio inter x et y inde resultans ad plurimas dimensiones assurgeret, qui tamen labor nihil aliud esset praestaturus, nisi ut ordinem, ad quem has curvas referri oportet, assignare valeamus. Caeterum quia hic duae quantitates arbitrariae A et B sunt introductae, evidens est iam innumerabiles lineas curvas diversas in hac sola solutione contineri.

10. Quo formulas has satis complicatas exemplo illustremus, ponamus $A=0$ et $B=1$ ac perveniemus ad sequentes formulas concinniores

$$x = \frac{2}{3}v\sqrt{v} + \frac{2}{15}(1-v)^{\frac{3}{2}}(3v+2)$$

et

$$y = -\frac{2}{3}(1-v)^{\frac{3}{2}} + \frac{2}{5}vv\sqrt{v}.$$

Quodsi hic loco $\frac{15}{2}x$ et $\frac{15}{2}y$ scribamus X et Y, quandoquidem hoc modo natura curvae non mutatur, tum vero eliminemus terminum $(1-v)^{\frac{3}{2}}$, pervenietur ad hanc aequationem

$$5X + Y(3v+2) = (25 + 6v + 9vv)v\sqrt{v},$$

quae aequatio denuo quadrari deberet ad rationalem efficiendam; tum vero littera v ascensura esset ad potestatem septimam, unde certe nemo determinationem huius litterae suscipiet.

EVOLUTIO CASUS QUO $U = \sqrt{v}$

11. Hic igitur occurrent binae sequentes formulae integrandae

$$\int \partial v \sqrt{(\alpha + \beta\sqrt{v})} \quad \text{et} \quad \int v\partial v \sqrt{(\alpha + \beta\sqrt{v})},$$

quas mox patebit itidem esse integrabiles. Si enim ponatur

$$\sqrt{(\alpha + \beta\sqrt{v})} = t,$$

erit

$$\sqrt{v} = \frac{tt - \alpha}{\beta},$$

consequenter

$$v = \frac{t^4 - 2\alpha tt + \alpha\alpha}{\beta\beta}, \quad \text{ergo} \quad \partial v = \frac{4t^3\partial t - 4\alpha t\partial t}{\beta\beta}.$$

Prior forma abibit in hanc $\frac{4tt\partial t}{\beta\beta}(tt - \alpha)$, cuius integrale est $\frac{4t^5}{5\beta\beta} - \frac{4\alpha t^3}{3\beta\beta}$, quamobrem habemus

$$\int \partial v \sqrt{(\alpha + \beta\sqrt{v})} = \frac{4(\alpha + \beta\sqrt{v})^{\frac{3}{2}}}{15\beta\beta}(3\beta\sqrt{v} - 2\alpha).$$

Pro altera autem formula habemus

$$v\partial v = \frac{(4t^7 - 12\alpha t^5 + 12\alpha\alpha t^3 - 4\alpha^3 t)\partial t}{\beta^4},$$

unde colligitur

$$\int v\partial v \sqrt{(\alpha + \beta\sqrt{v})} = \frac{4t^9}{9\beta^4} - \frac{12\alpha t^7}{7\beta^4} + \frac{12\alpha\alpha t^5}{5\beta^4} - \frac{4\alpha^3 t^3}{3\beta^4}.$$

Haec autem formula iam nimis est complicata, quam ut operae pretium foret

loco t eius valorem restituere; multo minus deinceps quisquam laborem esset susccepturus istas formulas integrales ad valores coordinatarum x et y transferendi.

12. Hic igitur nobis sufficiet ostendisse etiam hoc casu curvas prodituras esse algebraicas, quod iam porro sponte elucebit pro sequentibus casibus $U=\sqrt[3]{v}$, $U=\sqrt[4]{v}$ atque in genere $U=\sqrt[i]{v}$, quo casu posito $\sqrt{(\alpha+\beta\sqrt[i]{v})}=t$ erit $\sqrt[i]{v}=\frac{tt-\alpha}{\beta}$ ideoque $v=\left(\frac{tt-\alpha}{\beta}\right)^i$, ita ut v sit functio rationalis integra ipsius t, dummodo exponens i fuerit positivus et integer. Integratio igitur istarum formularum semper erit in potestate; quocirca etiam omnes isti casus perpetuo valores algebraicos pro coordinatis x et y suppeditabunt.

ALIA SOLUTIO PER ANGULOS INSTITUENDA

13. Utemur hic posterioribus formulis § 4 traditis, ubi ob $P=1$ et $Q=v$ habebimus

$$\partial x=\partial v\sin.\varphi+v\partial v\cos.\varphi \quad \text{et} \quad \partial y=\partial v\cos.\varphi-v\partial v\sin.\varphi.$$

Hic scilicet requiritur, ut eiusmodi angulus φ exploretur, quo istae formulae evadant integrabiles. Hoc facillime praestabitur statuendo $v=\sin.\theta$, ut sit $\partial v=\partial\theta\cos.\theta$, quo facto erit

$$\partial x=\partial\theta\cos.\theta\sin.\varphi+\partial\theta\sin.\theta\cos.\theta\cos.\varphi$$

et

$$\partial y=\partial\theta\cos.\theta\cos.\varphi-\partial\theta\sin.\theta\cos.\theta\sin.\varphi.$$

Est vero

$$\sin.\theta\cos.\theta=\frac{1}{2}\sin.2\theta,$$

$$\sin.p\cos.q=\frac{1}{2}\sin.(p+q)+\frac{1}{2}\sin.(p-q),$$

$$\sin.p\sin.q=\frac{1}{2}\cos.(p-q)-\frac{1}{2}\cos.(p+q),$$

$$\cos.p\cos.q=\frac{1}{2}\cos.(p+q)+\frac{1}{2}\cos.(p-q).$$

His igitur reductionibus in subsidium vocatis reperiemus

$$\frac{\partial x}{\partial\theta}=\frac{1}{2}\sin.(\varphi+\theta)+\frac{1}{2}\sin.(\varphi-\theta)+\frac{1}{4}\sin.(2\theta+\varphi)+\frac{1}{4}\sin.(2\theta-\varphi),$$

$$\frac{\partial y}{\partial\theta}=\frac{1}{2}\cos.(\varphi-\theta)+\frac{1}{2}\cos.(\varphi+\theta)-\frac{1}{4}\cos.(2\theta-\varphi)+\frac{1}{4}\cos.(2\theta+\varphi).$$

14. Iam vero evidens est singulas has partes integrationem esse admissuras, si modo anguli φ et θ rationem inter se teneant rationalem. Sit igitur $\varphi = \lambda\theta$ existente λ numero quocunque, sive integro sive fracto, sive positivo sive negativo; quin etiam generalius statui poterit $\varphi = \lambda\theta + \alpha$, quo facto habebimus

$$\frac{\partial x}{\partial \theta} = \frac{1}{2}\sin.[(\lambda+1)\theta+\alpha] + \frac{1}{2}\sin.[(\lambda-1)\theta+\alpha]$$
$$+\frac{1}{4}\sin.[(\lambda+2)\theta+\alpha] - \frac{1}{4}\sin.[(\lambda-2)\theta+\alpha],$$
$$\frac{\partial y}{\partial \theta} = \frac{1}{2}\cos.[(\lambda-1)\theta+\alpha] + \frac{1}{2}\cos.[(\lambda+1)\theta+\alpha]$$
$$-\frac{1}{4}\cos.[(\lambda-2)\theta+\alpha] + \frac{1}{4}\cos.[(\lambda+2)\theta+\alpha];$$

tum autem integratio nobis praebebit istas expressiones

$$x = -\frac{\cos.[(\lambda+1)\theta+\alpha]}{2(\lambda+1)} - \frac{\cos.[(\lambda-1)\theta+\alpha]}{2(\lambda-1)} - \frac{\cos.[(\lambda+2)\theta+\alpha]}{4(\lambda+2)} + \frac{\cos.[(\lambda-2)\theta+\alpha]}{4(\lambda-2)},$$
$$y = +\frac{\sin.[(\lambda-1)\theta+\alpha]}{2(\lambda-1)} + \frac{\sin.[(\lambda+1)\theta+\alpha]}{2(\lambda+1)} - \frac{\sin.[(\lambda-2)\theta+\alpha]}{4(\lambda-2)} + \frac{\sin.[(\lambda+2)\theta+\alpha]}{4(\lambda+2)},$$

quae formulae semper ergo erunt algebraicae, nisi fuerit vel $\lambda = \pm 1$ vel $\lambda = \pm 2$.

15. Consideremus casum, quo $\lambda = \frac{1}{2}$ et $\alpha = 0$, ac reperietur

$$x = \cos.\frac{1}{2}\theta - \frac{1}{2}\cos.\frac{3}{2}\theta - \frac{1}{10}\cos.\frac{5}{2}\theta$$

et

$$y = \sin.\frac{1}{2}\theta + \frac{1}{6}\sin.\frac{3}{2}\theta + \frac{1}{10}\sin.\frac{5}{2}\theta.$$

Porro cum sit

$$\sin.\frac{1}{2}\theta = \sin.\frac{3}{2}\theta\cos.\theta - \cos.\frac{3}{2}\theta\sin.\theta,$$
$$\cos.\frac{1}{2}\theta = \cos.\frac{3}{2}\theta\cos.\theta + \sin.\frac{3}{2}\theta\sin.\theta,$$
$$\sin.\frac{5}{2}\theta = \sin.\frac{3}{2}\theta\cos.\theta + \cos.\frac{3}{2}\theta\sin.\theta,$$
$$\cos.\frac{5}{2}\theta = \cos.\frac{3}{2}\theta\cos.\theta - \sin.\frac{3}{2}\theta\sin.\theta,$$

his valoribus substitutis habebimus

$$x = \frac{9}{10}\cos.\frac{3}{2}\theta\cos.\theta + \frac{11}{10}\sin.\frac{3}{2}\theta\sin.\theta - \frac{1}{2}\cos.\frac{3}{2}\theta$$

et

$$y = \frac{11}{10}\sin.\frac{3}{2}\theta\cos.\theta - \frac{9}{10}\cos.\frac{3}{2}\theta\sin.\theta + \frac{1}{6}\sin.\frac{3}{2}\theta.$$

Interim tamen et hic calculo satis taedioso foret opus, si hinc aequationem inter x et y elicere vellemus.

16. Evidens est hinc pariter innumerabiles inveniri lineas curvas problemati satisfacientes, quoniam litteras α et λ in infinitum variare licet. Utrum autem omnes istae solutiones a praecedentibus sint diversae necne, quaestio est altioris indaginis; in priori enim methodo variae solutiones deductae sunt ex variis formulis radicalibus $\sqrt{v}$, $\sqrt[3]{v}$, $\sqrt[4]{v}$, dum in posteriori petitae sunt ex multiplicatione seu divisione angulorum. Nulla autem affinitas inter has diversas determinationes intercedere videtur; atque adeo vix ullum est dubium, quin in linearum ordinibus inferioribus nullae plane dentur eiusmodi curvae, quarum arcus per arcus parabolicos exprimere liceat.

ADHUC ALIA SOLUTIO EIUSDEM PROBLEMATIS

17. Ponamus hic statim $v = \sin.\theta$, ut formula nostra adimplenda sit

$$\sqrt{(\partial x^2 + \partial y^2)} = \partial\theta\cos.\theta\sqrt{(1 + \sin.\theta^2)} = \partial\theta\cos.\theta\sqrt{(\cos.\theta^2 + 2\sin.\theta^2)}.$$

Faciamus $P = \cos.\theta$ et $Q = \sin.\theta\cdot\sqrt{2} = n\sin.\theta$ existente $\partial v = \partial\theta\cos.\theta$ et nunc ex § 4 habebimus

$$\frac{\partial x}{\partial\theta\cos.\theta} = \cos.\theta\sin.\varphi + n\sin.\theta\cos.\varphi$$

et

$$\frac{\partial y}{\partial\theta\cos.\theta} = \cos.\theta\cos.\varphi - n\sin.\theta\sin.\varphi,$$

quae aequationes in $\cos.\theta$ ductae ob

$$\cos.\theta^2 = \frac{1}{2} + \frac{1}{2}\cos.2\theta \quad \text{et} \quad \sin.\theta\cos.\theta = \frac{1}{2}\sin.2\theta$$

abeunt in istas

$$\frac{2\partial x}{\partial \theta} = \sin.\varphi + \cos.2\theta\sin.\varphi + n\sin.2\theta\cos.\varphi$$

et

$$\frac{2\partial y}{\partial \theta} = \cos.\varphi + \cos.2\theta\cos.\varphi - n\sin.2\theta\sin.\varphi,$$

hae autem porro ob

$$\cos.2\theta\sin.\varphi = \frac{1}{2}\sin.(\varphi + 2\theta) + \frac{1}{2}\sin.(\varphi - 2\theta)$$

et

$$\sin.2\theta\cos.\varphi = \frac{1}{2}\sin.(\varphi + 2\theta) - \frac{1}{2}\sin.(\varphi - 2\theta),$$

$$\cos.2\theta\cos.\varphi = \frac{1}{2}\cos.(\varphi + 2\theta) + \frac{1}{2}\cos.(\varphi - 2\theta)$$

et

$$\sin.2\theta\sin.\varphi = \frac{1}{2}\cos.(\varphi - 2\theta) - \frac{1}{2}\cos.(\varphi + 2\theta)$$

transformabuntur in sequentes

$$\frac{4\partial x}{\partial \theta} = 2\sin.\varphi + (n+1)\sin.(\varphi + 2\theta) - (n-1)\sin.(\varphi - 2\theta)$$

et

$$\frac{4\partial y}{\partial \theta} = 2\cos.\varphi + (n+1)\cos.(\varphi + 2\theta) - (n-1)\cos.(\varphi - 2\theta).$$

18. Nunc ambae istae formulae sponte integrabiles reddentur, si modo statuatur $\varphi = \alpha + \lambda\theta$; tum enim pro coordinatis curvae quaesitae habebimus

$$4x = -\frac{2}{\lambda}\cos.(\alpha + \lambda\theta) - \frac{n+1}{\lambda+2}\cos.(\alpha + (\lambda+2)\theta) + \frac{n-1}{\lambda-2}\cos.(\alpha + (\lambda-2)\theta),$$

$$4y = +\frac{2}{\lambda}\sin.(\alpha + \lambda\theta) + \frac{n+1}{\lambda+2}\sin.(\alpha + (\lambda+2)\theta) - \frac{n-1}{\lambda-2}\sin.(\alpha + (\lambda-2)\theta).$$

Hae igitur ambae formulae erunt algebraicae, dummodo ne sit vel $\lambda = 2$ vel $\lambda = -2$; reliquis casibus omnibus, quibus λ est numerus rationalis, sive integer sive fractus, curva prodibit algebraica.

19. Hic ergo sine dubio casus elicietur simplicissimus, si capiatur $\lambda = 1$ et $\alpha = 0$; tum enim habebimus

$$4x = -(n+1)\cos.\theta - \frac{n+1}{3}\cos.3\theta$$

et

$$4y = -(n-3)\sin.\theta + \frac{n+1}{3}\sin.3\theta,$$

ubi litteram n scripsimus loco $\sqrt{2}$.

20. Ad has formulas tractandas ponamus $\operatorname{tang}.\theta = t$ fietque

$$\sin.\theta = \frac{t}{\sqrt{(1+tt)}} \quad \text{et} \quad \cos.\theta = \frac{1}{\sqrt{(1+tt)}},$$

tum vero erit $\operatorname{tang}.3\theta = \frac{3t-t^3}{1-3tt}$, unde fit

$$\sin.3\theta = \frac{3t-t^3}{(1+tt)^{\frac{3}{2}}} \quad \text{et} \quad \cos.3\theta = \frac{1-3tt}{(1+tt)^{\frac{3}{2}}},$$

quibus valoribus substitutis reperiemus

$$-4x = \frac{+(n+1)}{\sqrt{(1+tt)}} + \frac{(n+1)(1-3tt)}{3(1+tt)^{\frac{3}{2}}} = \frac{4(n+1)}{3(1+tt)^{\frac{3}{2}}}$$

ideoque

$$x = \frac{-(n+1)}{3(1+tt)^{\frac{3}{2}}} \quad \text{et} \quad y = \frac{t}{3(1+tt)^{\frac{3}{2}}}(3-(n-2)tt).$$

Dividatur posterior aequatio per priorem et prodibit

$$\frac{(n+1)y}{x} = (n-2)t^3 - 3t.$$

Hinc autem satis liquet, si vellemus quantitatem t eliminare, aequationem inter x et y ad plurimas dimensiones esse adscensuram. Sufficiat igitur tres formulas generales exhibuisse, quarum singulae innumerabiles curvas algebraicas suppeditare possunt, ita ut in omnibus longitudo arcus curvae $\int\sqrt{(\partial x^2+\partial y^2)}$ aequetur arcui parabolico $\int\partial v\sqrt{(1+vv)}$.

DE INNUMERIS CURVIS ALGEBRAICIS QUARUM LONGITUDINEM PER ARCUS ELLIPTICOS METIRI LICET

Convent. exhib. die 10 Iunii 1776

Commentatio 639 indicis ENESTROEMIANI

Nova acta academiae scientiarum Petropolitanae 5 (1787), 1789, p. 71—85

Summarium ibidem p. 67—69

SUMMARIUM

Après avoir réussi si bien, dans le Mémoire précédent, à trouver une infinité de courbes algébriques dont la longueur pût être mesurée par des arcs paraboliques, il étoit tout naturel d'essayer aussi les arcs elliptiques. Tout revenoit à trouver pour l'abscisse x et l'ordonnée y des fonctions de v telles que

$$\sqrt{(\partial x^2 + \partial y^2)} = \frac{\partial v \sqrt{[1 + (nn-1)vv]}}{\sqrt{(1 - vv)}},$$

formule qui exprime un arc elliptique dont l'abscisse est v et l'ordonnée $n\sqrt{(1 - vv)}$.

L'Auteur donne de ce problème, comme de celui du Mémoire précédent, trois solutions différentes, qui mènent chacune à une infinité de courbes algébriques mesurables par des arcs elliptiques. Nous allons encore présenter au lecteur l'esprit de la troisième, comme de la plus courte.

Soit $v = \sin. \varphi$, de façon que

$$\sqrt{(\partial x^2 + \partial y^2)} = \partial \varphi \sqrt{[1 + (nn - 1) \sin. \varphi^2]} = \partial \varphi \sqrt{(1 + m^2 \sin. \varphi^2)}.$$

A cette équation satisfont les valeurs

$$\partial x = \partial \varphi \cos. \lambda \varphi - m \partial \varphi \sin. \varphi \sin. \lambda \varphi,$$
$$\partial y = \partial \varphi \sin. \lambda \varphi + m \partial \varphi \sin. \varphi \cos. \lambda \varphi,$$

qui peuvent aussi être représentées ainsi

$$\partial x = \frac{1}{2}\partial\varphi[2\cos.\lambda\varphi - m\cos.(\lambda-1)\varphi + m\cos.(\lambda+1)\varphi],$$

$$\partial y = \frac{1}{2}\partial\varphi[2\sin.\lambda\varphi + m\sin.(\lambda+1)\varphi - m\sin.(\lambda-1)\varphi],$$

dont les intégrales fournissent les abscisses et ordonnées d'une infinité de courbes algébriques qui satisfont à la condition du problème.

M. EULER observe que, quoique le nombre des solutions qu'il a données des problèmes qui font le sujet des deux derniers mémoires soit infini, on ne sauroit soutenir que ces formules épuisent toutes les solutions possibles. L'Auteur avoit essayé aussi plus d'une fois de chercher des courbes algébriques qui pussent être mesurés par des portions d'hyperbole, mais il n'en a jamais pu trouver une seule. Cependant il n'oseroit soutenir qu'il n'y en eût pas, comme il avoit fait avec assurance à l'égard du cercle; et il invite les géomètres, à la fin de son Mémoire, à s'occuper d'un sujet d'analyse qui paroît promettre une riche récolte de vérités nouvelles et intéressantes.

1. Pro ellipsi, cuius singuli arcus nobis mensuram curvarum quaesitarum suppeditare debent, sit abscissa $= v$, applicata vero $= n\sqrt{(1-vv)}$, unde elementum arcus colligitur $= \frac{\partial v\sqrt{[1+(nn-1)vv]}}{\sqrt{(1-vv)}}$; quamobrem sequens nobis propositum sit problema.

PROBLEMA

Pro coordinatis x et y eiusmodi functiones algebraicas ipsius v investigare, ut fiat

$$\sqrt{(\partial x^2 + \partial y^2)} = \frac{\partial v\sqrt{[1+(nn-1)vv]}}{\sqrt{(1-vv)}}.$$

SOLUTIO

2. Ut formulae $\sqrt{(\partial x^2+\partial y^2)}$ formam praescriptam conciliemus, quoniam denominator $\sqrt{(1-vv)}$ duos habet factores $\sqrt{(1+v)}$ et $\sqrt{(1-v)}$, statuamus

$$\partial x = \frac{(p+q)\partial v}{\sqrt{2(1+v)}}, \qquad \partial y = \frac{(p-q)\partial v}{\sqrt{2(1-v)}};$$

hinc autem fiet

$$\sqrt{(\partial x^2 + \partial y^2)} = \frac{\partial v \sqrt{(pp + qq - 2pqv)}}{\sqrt{(1 - vv)}},$$

unde patet pro p et q eiusmodi quantitates quaeri debere, ut prodeat

$$pp + qq - 2pqv = 1 + (nn - 1)vv.$$

3. Ante omnia autem hic evidens est, si modo pro litteris p et q functiones rationales integrae ipsius v assignari queant, ambas formulas pro ∂x et ∂y assumtas semper integrationem esse admissuras, propterea quod ambae istae formulae

$$\frac{v^i \partial v}{\sqrt{(1 + v)}} \quad \text{et} \quad \frac{v^i \partial v}{\sqrt{(1 - v)}}$$

semper sunt integrabiles, si modo exponens i fuerit integer positivus. Ad hoc igitur negotium absolvendum sequentes casus evolvamus.

I. CASUS QUO $p = 1$ ET $q = \alpha v$

4. Hic igitur erit

$$pp + qq = 1 + \alpha\alpha vv \quad \text{et} \quad 2pqv = 2\alpha vv,$$

quamobrem effici oportet

$$1 + \alpha\alpha vv - 2\alpha vv = 1 + (nn - 1)vv,$$

unde patet sumi debere $\alpha = 1 + n$, ita ut nostra elementa hoc casu fiant

$$\partial x = \frac{[1 + (n + 1)v]\partial v}{\sqrt{2(1 + v)}} \quad \text{et} \quad \partial y = \frac{[1 - (n + 1)v]\partial v}{\sqrt{2(1 - v)}},$$

ubi integralibus sumtis reperitur

$$x = \frac{1}{3}[1 - 2n + (n + 1)v]\sqrt{2(1 + v)}$$

et

$$y = \frac{1}{3}[2n - 1 + (n + 1)v]\sqrt{2(1 - v)}.$$

5. Ut hinc quantitatem v eliminemus, addamus ambo quadrata et obtinebimus

$$\frac{9(xx+yy)}{4}=(2n-1)^2-3(nn-1)vv,$$

ex qua aequatione v facile per x et y determinatur; inde enim fit

$$vv=\frac{(2n-1)^2}{3(nn-1)}-\frac{3(xx+yy)}{4(nn-1)}.$$

Quo iam hunc valorem loco vv facilius substituere queamus, sumamus productum nostrarum formularum

$$\frac{9xy}{2}=[(n+1)^2vv-(2n-1)^2]\sqrt{(1-vv)},$$

quae aequatio si quadretur, ubique tantum pares dimensiones ipsius v occurrent ac loco vv valore substituto aequatio inter x et y ad sextum ordinem ascendet.

II. CASUS QUO $p=1+\beta vv$ ET $q=\alpha v$

6. Hic ergo erit

$$pp+qq=1+(\alpha\alpha+2\beta)vv+\beta\beta v^4$$

et

$$2pqv=2\alpha vv+2\alpha\beta v^4,$$

unde conditio adimplenda erit

$$1+(\alpha\alpha+2\beta-2\alpha)vv+(\beta\beta-2\alpha\beta)v^4=1+(nn-1)vv.$$

Hic igitur ante omnia esse oportet

$$\beta\beta-2\alpha\beta=0 \quad \text{ideoque} \quad \beta=2\alpha$$

atque nunc superest, ut fiat

$$\alpha\alpha+2\beta-2\alpha=\alpha\alpha+2\alpha=nn-1,$$

sicque capi debet

$$\alpha=n-1 \quad \text{et} \quad \beta=2(n-1).$$

7. Pro curva igitur definienda habebimus

$$p = 1 + 2(n-1)vv \quad \text{et} \quad q = (n-1)v$$

sicque nunc erit

$$\partial x = \frac{1 + (n-1)v + 2(n-1)vv}{\sqrt{2(1+v)}}\,\partial v \quad \text{et} \quad \partial y = \frac{1 - (n-1)v + 2(n-1)vv}{\sqrt{2(1-v)}}\,\partial v,$$

quarum integratio nulla amplius laborat difficultate, unde hoc labore merito supersedemus.

III. CASUS QUO $p = 1 + \beta vv$ ET $q = \alpha v + \gamma v^3$

8. Hic igitur erit

$$pp + qq = 1 + (\alpha\alpha + 2\beta)vv + (\beta\beta + 2\alpha\gamma)v^4 + \gamma\gamma v^6$$

et

$$pq = \alpha v + (\alpha\beta + \gamma)v^3 + \beta\gamma v^5,$$

unde conficitur

$$pp + qq - 2pqv$$
$$= 1 + (\alpha\alpha + 2\beta - 2\alpha)vv + (\beta\beta + 2\alpha\gamma - 2\alpha\beta - 2\gamma)v^4 + (\gamma\gamma - 2\beta\gamma)v^6,$$

quae quantitas aequari debet $1 + (nn-1)vv$. Hic igitur primo potestas v^6 tolli debet, quod fit ponendo

$$\gamma\gamma - 2\beta\gamma = 0 \quad \text{ideoque} \quad \gamma = 2\beta;$$

deinde vero etiam potestatem quartam tolli oportet, unde fit

$$\beta\beta + 2\alpha\gamma - 2\alpha\beta - 2\gamma = 0 \quad \text{sive} \quad \beta\beta + 2\alpha\beta - 4\beta = 0$$

ideoque

$$\beta = 4 - 2\alpha \quad \text{et} \quad \gamma = 8 - 4\alpha.$$

Iam vero coefficiens ipsius vv erit

$$\alpha\alpha + 2\beta - 2\alpha = \alpha\alpha + 8 - 6\alpha,$$

quem aequari oportet ipsi $nn - 1$, unde colligitur $\alpha - 3 = n$ sive

$$\alpha = n + 3,$$

tum vero

$$\beta = -2(n+1) \quad \text{et} \quad \gamma = -4(n+1).$$

5. Ut hinc quantitatem v eliminemus, addamus ambo quadrata et obtinebimus

$$\frac{9(xx+yy)}{4} = (2n-1)^2 - 3(nn-1)vv,$$

ex qua aequatione v facile per x et y determinatur; inde enim fit

$$vv = \frac{(2n-1)^2}{3(nn-1)} - \frac{3(xx+yy)}{4(nn-1)}.$$

Quo iam hunc valorem loco vv facilius substituere queamus, sumamus productum nostrarum formularum

$$\frac{9xy}{2} = [(n+1)^2 vv - (2n-1)^2]\sqrt{(1-vv)},$$

quae aequatio si quadretur, ubique tantum pares dimensiones ipsius v occurrent ac loco vv valore substituto aequatio inter x et y ad sextum ordinem ascendet.

II. CASUS QUO $p = 1 + \beta vv$ ET $q = \alpha v$

6. Hic ergo erit

$$pp + qq = 1 + (\alpha\alpha + 2\beta)vv + \beta\beta v^4$$

et

$$2pqv = 2\alpha vv + 2\alpha\beta v^4,$$

unde conditio adimplenda erit

$$1 + (\alpha\alpha + 2\beta - 2\alpha)vv + (\beta\beta - 2\alpha\beta)v^4 = 1 + (nn-1)vv.$$

Hic igitur ante omnia esse oportet

$$\beta\beta - 2\alpha\beta = 0 \quad \text{ideoque} \quad \beta = 2\alpha$$

atque nunc superest, ut fiat

$$\alpha\alpha + 2\beta - 2\alpha = \alpha\alpha + 2\alpha = nn - 1,$$

sicque capi debet

$$\alpha = n - 1 \quad \text{et} \quad \beta = 2(n-1).$$

7. Pro curva igitur definienda habebimus

$$p = 1 + 2(n-1)vv \quad \text{et} \quad q = (n-1)v$$

sicque nunc erit

$$\partial x = \frac{1 + (n-1)v + 2(n-1)vv}{\sqrt{2(1+v)}}\partial v \quad \text{et} \quad \partial y = \frac{1-(n-1)v + 2(n-1)vv}{\sqrt{2(1-v)}}\partial v,$$

quarum integratio nulla amplius laborat difficultate, unde hoc labore merito supersedemus.

III. CASUS QUO $p = 1 + \beta vv$ ET $q = \alpha v + \gamma v^3$

8. Hic igitur erit

$$pp + qq = 1 + (\alpha\alpha + 2\beta)vv + (\beta\beta + 2\alpha\gamma)v^4 + \gamma\gamma v^6$$

et

$$pq = \alpha v + (\alpha\beta + \gamma)v^3 + \beta\gamma v^5,$$

unde conficitur

$$pp + qq - 2pqv$$
$$= 1 + (\alpha\alpha + 2\beta - 2\alpha)vv + (\beta\beta + 2\alpha\gamma - 2\alpha\beta - 2\gamma)v^4 + (\gamma\gamma - 2\beta\gamma)v^6,$$

quae quantitas aequari debet $1 + (nn-1)vv$. Hic igitur primo potestas v^6 tolli debet, quod fit ponendo

$$\gamma\gamma - 2\beta\gamma = 0 \quad \text{ideoque} \quad \gamma = 2\beta;$$

deinde vero etiam potestatem quartam tolli oportet, unde fit

$$\beta\beta + 2\alpha\gamma - 2\alpha\beta - 2\gamma = 0 \quad \text{sive} \quad \beta\beta + 2\alpha\beta - 4\beta = 0$$

ideoque

$$\beta = 4 - 2\alpha \quad \text{et} \quad \gamma = 8 - 4\alpha.$$

Iam vero coefficiens ipsius vv erit

$$\alpha\alpha + 2\beta - 2\alpha = \alpha\alpha + 8 - 6\alpha,$$

quem aequari oportet ipsi $nn-1$, unde colligitur $\alpha - 3 = n$ sive

$$\alpha = n + 3,$$

tum vero

$$\beta = -2(n+1) \quad \text{et} \quad \gamma = -4(n+1).$$

9. His igitur valoribus inventis nostrae formulae integrandae erunt

$$\partial x = \frac{1+(n+3)v-2(n+1)vv-4(n+1)v^3}{\sqrt{2(1+v)}}\partial v$$

et

$$\partial y = \frac{1-(n+3)v-2(n+1)vv+4(n+1)v^3}{\sqrt{2(1-v)}}\partial v,$$

quarum integrationi iterum non immorabimur. Unicum tantum adhuc talem casum attingamus.

IV. CASUS QUO $p = 1+\beta vv+\delta v^4$ ET $q = \alpha v+\gamma v^3$

10. Hic igitur erit

$$pp+qq = 1+(\alpha\alpha+2\beta)vv+(\beta\beta+2\delta+2\alpha\gamma)v^4+(2\beta\delta+\gamma\gamma)v^6+\delta\delta v^8$$

et

$$pq = \alpha v+(\alpha\beta+\gamma)v^3+(\alpha\delta+\beta\gamma)v^5+\gamma\delta v^7,$$

ex quibus conficitur formula

$$\begin{aligned}pp+qq-2pqv = 1&+(\alpha\alpha+2\beta-2\alpha)vv+(\beta\beta+2\alpha\gamma+2\delta-2\alpha\beta-2\gamma)v^4\\&+(\gamma\gamma+2\beta\delta-2\alpha\delta-2\beta\gamma)v^6+(\delta\delta-2\gamma\delta)v^8;\end{aligned}$$

quae formula cum aequari debeat huic $1+(nn-1)vv$, primo tollatur potestas octava, unde fit

$$\delta\delta-2\gamma\delta = 0 \quad \text{ideoque} \quad \delta = 2\gamma.$$

Iam potestas sexta afficitur hac forma

$$\gamma\gamma+2\beta\delta-2\alpha\delta-2\beta\gamma = \gamma\gamma+2\beta\gamma-4\alpha\gamma,$$

quae nihilo aequata praebet

$$\gamma = 4\alpha-2\beta \quad \text{hincque} \quad \delta = 8\alpha-4\beta.$$

Porro autem potestatis quartae coefficiens est

$$\beta\beta+2\alpha\gamma+2\delta-2\alpha\beta-2\gamma = \beta\beta-6\alpha\beta-4\beta+8\alpha\alpha+8\alpha = 0,$$

quae aequatio divisa per $\beta - 2\alpha$ praebet $\beta - 4\alpha - 4$, ita ut pro β geminos nanciscamur valores, alterum $\beta = 2\alpha$, alterum vero $\beta = 4\alpha + 4$, quorum utrumque seorsim evolvamus.

11. Sit igitur $\beta = 2\alpha$ eritque $\gamma = 0$ et $\delta = 0$, quo ergo casu res ad casum secundum revolvitur. Sit igitur

$$\beta = 4\alpha + 4$$

et nunc fiet

$$\gamma = -4(\alpha + 2) \quad \text{et} \quad \delta = -8(\alpha + 2).$$

Verum hinc fiet potestatis vv coefficiens $\alpha\alpha + 2\beta - 2\alpha$ ipsi $nn - 1$ aequandus, unde fit $\alpha + 3 = n$ sive

$$\alpha = n - 3,$$

hincque porro fiet

$$\beta = 4(n-2), \quad \gamma = -4(n-1) \quad \text{et} \quad \delta = -8(n-1),$$

ex quibus ergo conficitur

$$p = 1 + 4(n-2)vv - 8(n-1)v^4 \quad \text{et} \quad q = (n-3)v - 4(n-1)v^3,$$

unde tandem colligitur

$$x = \int \frac{(p+q)\partial v}{\sqrt{2(1+v)}} \quad \text{et} \quad y = \int \frac{(p-q)\partial v}{\sqrt{2(1-v)}},$$

quem integrationis laborem suscipere foret superfluum.

DIGRESSIO PRO CASU $n = \pm 1$

12. Ex evolutione casuum superiorum manifestum est curvas continuo ad altiores gradus ascendere; hinc autem perpetuo excipi oportet casum, quo foret $n = \pm 1$, quandoquidem arcus ellipticus $\int \frac{\partial v \sqrt{[1 + (nn-1)vv]}}{\sqrt{(1-vv)}}$ abiret in $\int \frac{\partial v}{\sqrt{(1-vv)}}$, hoc est in arcum circularem. Cum igitur praeter circulum nullae aliae dentur tales curvae[1]), necesse est, ut curvae, ad quos casus praecedentes nos deducunt, quando fuerit $n = \pm 1$, circulum exhibeant.

1) Vide notam p. 83. A. K.

13. Pro casu autem primo, ubi integralia iam evolvimus, quando fit $n = \pm 1$ ideoque $nn - 1 = 0$, aequatio penultima abit in hanc

$$\frac{9}{4}(xx + yy) = (2n - 1)^2,$$

hoc est, aequabitur vel $= 1$ vel $= 9$, ita ut utroque casu curva manifesto sit circulus, cum tamen pro aliis omnibus valoribus ipsius n aequatio ad sextum ordinem assurgere sit observata.

14. Pro casu secundo faciamus primo $n = +1$ eritque

$$\partial x = \frac{\partial v.}{\sqrt{2(1+v)}} \quad \text{et} \quad \partial y = \frac{\partial v}{\sqrt{2(1-v)}},$$

unde integrando fit

$$x = \sqrt{2(1+v)} \quad \text{et} \quad y = -\sqrt{2(1-v)},$$

ex quibus manifesto colligitur

$$xx + yy = 4,$$

quae utique est aequatio ad circulum.

Sin autem sumamus $n = -1$, reperitur

$$\partial x = \frac{1 - 2v - 4vv}{\sqrt{2(1+v)}}\partial v \quad \text{et} \quad \partial y = \frac{1 + 2v - 4vv}{\sqrt{2(1-v)}}\partial v;$$

hinc autem circulum enasci sequenti modo facillime ostendetur.

15. Hunc in finem statuatur

$$v = \cos. 2\varphi$$

eritque $\partial v = -2\partial\varphi \sin. 2\varphi$ et $\sqrt{2(1+v)} = 2 \cos. \varphi$ similique modo $\sqrt{2(1-v)} = 2 \sin. \varphi$. Ergo pro priore formula erit

$$\frac{\partial v}{\sqrt{2(1+v)}} = -2\partial\varphi \sin. \varphi,$$

alter vero factor fiet

$$= 1 - 2 \cos. 2\varphi - 4 \cos. 2\varphi^2 = -1 - 2 \cos. 2\varphi - 2 \cos. 4\varphi,$$

quamobrem habebimus

$$\partial x = 2\partial\varphi \sin.\varphi(1 + 2\cos.2\varphi + 2\cos.4\varphi).$$

Constat autem esse

$$2\sin.\varphi\cos.2\varphi = \sin.3\varphi - \sin.\varphi \quad \text{et} \quad 2\sin.\varphi\cos.4\varphi = \sin.5\varphi - \sin.3\varphi,$$

quibus substitutis obtinebitur $\partial x = 2\partial\varphi\sin.5\varphi$, cuius integrale est

$$x = -\frac{2}{5}\cos.5\varphi.$$

Simili modo pro altera formula prodit

$$\frac{\partial v}{\sqrt{2(1-v)}} = -2\partial\varphi\cos.\varphi,$$

alter vero factor erit

$$= 1 + 2\cos.2\varphi - 4\cos.2\varphi^2 = -1 + 2\cos.2\varphi - 2\cos.4\varphi$$

sicque fiet

$$\partial y = 2\partial\varphi\cos.\varphi(1 - 2\cos.2\varphi + 2\cos.4\varphi).$$

Constat autem esse

$$2\cos.\varphi\cos.2\varphi = \cos.3\varphi + \cos.\varphi \quad \text{et} \quad 2\cos.\varphi\cos.4\varphi = \cos.5\varphi + \cos.3\varphi,$$

quocirca proveniet $\partial y = 2\partial\varphi\cos.5\varphi$, ideoque

$$y = \frac{2}{5}\sin.5\varphi;$$

consequenter hic casus nobis suppeditat

$$xx + yy = \frac{4}{25},$$

quae iterum manifesto est aequatio ad circulum.

16. Tractemus simili modo casum tertium ponendo primo $n = +1$ et habebimus

$$\partial x = \frac{1 + 4v - 4vv - 8v^3}{\sqrt{2(1+v)}}\partial v \quad \text{et} \quad \partial y = \frac{1 - 4v - 4vv + 8v^3}{\sqrt{2(1-v)}}\partial v.$$

Statuamus iterum $v = \cos.2\varphi$ fietque

$$\partial x = -2\partial\varphi \sin.\varphi(1 + 4\cos.2\varphi - 4\cos.2\varphi^2 - 8\cos.2\varphi^3)$$

et

$$\partial y = -2\partial\varphi \cos.\varphi(1 - 4\cos.2\varphi - 4\cos.2\varphi^2 + 8\cos.2\varphi^3),$$

ubi notetur esse

$$4\cos.2\varphi^2 = 2 + 2\cos.4\varphi \quad \text{et} \quad 8\cos.2\varphi^3 = 6\cos.2\varphi + 2\cos.6\varphi,$$

unde habebimus

$$\partial x = 2\partial\varphi \sin.\varphi(1 + 2\cos.2\varphi + 2\cos.4\varphi + 2\cos.6\varphi)$$

et

$$\partial y = 2\partial\varphi \cos.\varphi(1 - 2\cos.2\varphi + 2\cos.4\varphi - 2\cos.6\varphi).$$

Quodsi iam has reductiones ulterius prosequamur, nanciscemur tandem

$$\partial x = 2\partial\varphi \sin.7\varphi \quad \text{ideoque} \quad x = -\frac{2}{7}\cos.7\varphi$$

eodemque modo

$$\partial y = 2\partial\varphi \cos.7\varphi, \quad \text{ergo} \quad y = \frac{2}{7}\sin.7\varphi,$$

unde iterum colligitur

$$xx + yy = \frac{4}{49}$$

ideoque pro circulo.

17. Si pro eodem casu tertio ponatur $n = -1$, fiet

$$\partial x = \frac{1+2v}{\sqrt{2(1+v)}}\partial v \quad \text{et} \quad \partial y = \frac{1-2v}{\sqrt{2(1-v)}}\partial v.$$

Statuamus igitur $v = \cos.2\varphi$ eritque

$$\sqrt{2(1+v)} = 2\cos.\varphi \quad \text{et} \quad \sqrt{2(1-v)} = 2\sin.\varphi$$

et

$$\partial v = -2\partial\varphi \sin.2\varphi;$$

quamobrem habebitur

$$\partial x = -\frac{1+2\cos.2\varphi}{\cos.\varphi}\partial\varphi\sin.2\varphi = -2\partial\varphi\sin.\varphi(1+2\cos.2\varphi)$$

et

$$\partial y = -\frac{1-2\cos.2\varphi}{\sin.\varphi}\partial\varphi\sin.2\varphi = -2\partial\varphi\cos.\varphi(1-2\cos.2\varphi),$$

quae formulae porro reducuntur ad has

$$\partial x = -2\partial\varphi\sin.3\varphi \quad \text{et} \quad \partial y = -2\partial\varphi\cos.3\varphi$$

hincque integrando fiet

$$x = \frac{2}{3}\cos.3\varphi \quad \text{et} \quad y = -\frac{2}{3}\sin.3\varphi,$$

unde colligitur

$$xx + yy = \frac{4}{9},$$

quae est aequatio pro circulo, cuius radius $= \frac{2}{3}$.

18. Quodsi quis simili modo casum quartum evolvere voluerit ponendo sive $n = +1$ sive $n = -1$, itidem reperiet curvas satisfacientes pariter ad circulum reduci. Hinc igitur ansam arripimus problema nostrum alio modo resolvendi, dum scilicet in formulam, qua arcus curvae exprimi debet, statim sinum cosinumve cuiuspiam anguli introducemus.

ALIA PROBLEMATIS SOLUTIO EX CALCULO ANGULORUM PETITA

19. Cum elementum arcus curvarum quaesitarum debeat esse

$$\partial s = \frac{\partial v\sqrt{[1+(nn-1)vv]}}{\sqrt{(1-vv)}},$$

ponamus statim $v = \sin.\varphi$, ut fiat $\frac{\partial v}{\sqrt{(1-vv)}} = \partial\varphi$, eritque

$$\partial s = \partial\varphi\sqrt{(\cos.\varphi^2 + nn\sin.\varphi^2)},$$

unde statim manifestum est capi posse

$$\partial x = \partial\varphi \cos.\varphi \quad \text{et} \quad \partial y = n\partial\varphi \sin.\varphi,$$

unde fit

$$x = \sin.\varphi \quad \text{et} \quad y = -n\cos.\varphi$$

ideoque $nx = n\sin.\varphi$, unde colligitur

$$nnxx + yy = nn,$$

quae est ipsa aequatio pro ellipsi, cuius arcus mensuram reliquarum curvarum constituere debent.

20. Ex hac autem solutione infinitas alias curvas quaesito pariter satisfacientes derivare possumus ponendo

$$\partial x = \partial\varphi \cos.\varphi \cos.\omega - n\partial\varphi \sin.\varphi \sin.\omega$$

et

$$\partial y = \partial\varphi \cos.\varphi \sin.\omega + n\partial\varphi \sin.\varphi \cos.\omega;$$

sic enim evadet

$$\partial x^2 + \partial y^2 = \partial\varphi^2 \cos.\varphi^2 + nn\partial\varphi^2 \sin.\varphi^2 = \partial s^2.$$

Tantum igitur superest, ut istae duae formulae differentiales integrabiles reddantur, quod manifesto in genere eveniet sumendo $\omega = \lambda\varphi$; tum enim per reductiones satis cognitas nanciscemur

$$\frac{2\partial x}{\partial\varphi} = (n+1)\cos.(\lambda+1)\varphi - (n-1)\cos.(\lambda-1)\varphi$$

et

$$\frac{2\partial y}{\partial\varphi} = (n+1)\sin.(\lambda+1)\varphi - (n-1)\sin.(\lambda-1)\varphi$$

atque hinc integrando impetrabimus

$$2x = \frac{n+1}{\lambda+1}\sin.(\lambda+1)\varphi - \frac{n-1}{\lambda-1}\sin.(\lambda-1)\varphi,$$

$$-2y = \frac{n+1}{\lambda+1}\cos.(\lambda+1)\varphi - \frac{n-1}{\lambda-1}\cos.(\lambda-1)\varphi,$$

quae ergo ambae formulae semper sunt algebraicae solo casu excepto, ubi $\lambda = \pm 1$. Caeterum quando $n = \pm 1$, curvae resultantes manifesto abeunt in circulum, quicunque valor ipsi λ tribuatur.

21. Haec solutio non solum est admodum succincta, sed etiam multo latius patet quam praecedens, quandoquidem praecedentes casus ex hac solutione deducuntur sumendo $\lambda = \pm\frac{1}{2}$ vel $\lambda = \pm\frac{3}{2}$ vel $\lambda = \pm\frac{5}{2}$ vel $\lambda = \pm\frac{7}{2}$. Quatenus igitur hic pro λ numeros integros accipere licet vel etiam quascunque alias fractiones, eatenus haec solutio longe alias suppeditat lineas curvas, quae ex priori solutione nullo modo deduci possunt. Evolvamus igitur aliquot exempla.

EXEMPLUM 1

22. Quia pro λ unitatem assumere non licet, ponamus statim $\lambda = 2$ atque habebimus

$$x = +\frac{n+1}{6}\sin. 3\varphi - \frac{n-1}{2}\sin. \varphi$$

et

$$y = -\frac{n+1}{6}\cos. 3\varphi + \frac{n-1}{2}\cos. \varphi;$$

hinc iam colligimus

$$xx + yy = \frac{(n+1)^2}{36} + \frac{(n-1)^2}{4} - \frac{nn-1}{6}\cos. 2\varphi,$$

ex qua aequatione angulus φ haud difficulter per x et y determinatur, qui deinceps in alterutra substitutus praebebit aequationem inter x et y.

EXEMPLUM 2

Sumamus etiam $\lambda = \frac{1}{2}$; erit

$$x = +\frac{n+1}{3}\sin. \frac{3}{2}\varphi - (n-1)\sin. \frac{1}{2}\varphi$$

et

$$y = -\frac{n+1}{3}\cos. \frac{3}{2}\varphi - (n-1)\cos. \frac{1}{2}\varphi.$$

Facile autem patet hoc exemplum cum casu supra § 4 tractato congruere.

SCHOLION

23. Haec igitur solutio praecedentem maxime supereminet, cum non solum infinities plures curvas in se complectatur, sed etiam valores pro coordinatis

x et y inventi tam simpliciter exprimantur, ut duobus tantum terminis constent, cum sit

$$2x = \frac{n+1}{\lambda+1}\sin.(\lambda+1)\varphi - \frac{n-1}{\lambda-1}\sin.(\lambda-1)\varphi$$

et

$$-2y = \frac{n+1}{\lambda+1}\cos.(\lambda+1)\varphi - \frac{n-1}{\lambda-1}\cos.(\lambda-1)\varphi.$$

Ex qua forma coordinatarum colligitur istas curvas omnes affines esse epicycloidibus et generari posse ex provolutione circuli super circulo, dum scilicet stylus describens non in ipsa peripheria circuli mobilis assumitur. Interim tamen ne haec quidem solutio pro generali haberi potest; namque innumerabiles alias curvas satisfacientes assignare licet, quae ne in hac quidem solutione continentur, quam inventionem hic subiungamus.

ADHUC ALIA SOLUTIO PROBLEMATIS PROPOSITI

24. Maneat ut ante $v = \sin.\varphi$, et cum hinc fiat

$$\partial s = \partial\varphi\sqrt{(\cos.\varphi^2 + nn\sin.\varphi^2)},$$

scribamus $1 - \sin.\varphi^2$ loco $\cos.\varphi^2$ eritque

$$\partial s = \partial\varphi\sqrt{(1 + (nn-1)\sin.\varphi^2)}.$$

Faciamus autem brevitatis gratia $nn - 1 = mm$ atque huic conditioni statim satisfieret sumendo $\partial x = \partial\varphi$ et $\partial y = m\partial\varphi\sin.\varphi$; hinc autem ob $x = \varphi$ prodiret curva transcendens, quod tamen non impedit, quominus infinitae curvae algebraicae hinc deduci queant. Statuamus enim

$$\partial x = \partial\varphi\cos.\lambda\varphi - m\partial\varphi\sin.\varphi\sin.\lambda\varphi$$

et

$$\partial y = \partial\varphi\sin.\lambda\varphi + m\partial\varphi\sin.\varphi\cos.\lambda\varphi$$

atque hinc prodit

$$\partial x^2 + \partial y^2 = \partial\varphi^2(1 + mm\sin.\varphi^2).$$

Nunc igitur membra posteriora more solito evolvantur et obtinebitur

$$\frac{2\partial x}{\partial\varphi} = 2\cos.\lambda\varphi - m\cos.(\lambda-1)\varphi + m\cos.(\lambda+1)\varphi$$

et

$$\frac{2\partial y}{\partial\varphi} = 2\sin.\lambda\varphi + m\sin.(\lambda+1)\varphi - m\sin.(\lambda-1)\varphi,$$

unde sumtis integralibus erit

$$2x = \frac{2}{\lambda}\sin.\lambda\varphi - \frac{m}{\lambda-1}\sin.(\lambda-1)\varphi + \frac{m}{\lambda+1}\sin.(\lambda+1)\varphi$$

et

$$-2y = \frac{2}{\lambda}\cos.\lambda\varphi - \frac{m}{\lambda-1}\cos.(\lambda-1)\varphi + \frac{m}{\lambda+1}\cos.(\lambda+1)\varphi,$$

quae ergo formulae etiam sunt algebraicae solo casu $\lambda = \pm 1$ excepto. Perspicuum autem est hos valores penitus esse diversos a praecedentibus, propterea quod terna membra involvunt.

25. Praeterea vero hic manifesto assumsimus esse $nn > 1$, ita ut haec solutio extendi nequeat ad casus, quibus $nn < 1$, cum prior solutio pro omnibus valoribus numeri n valeat; interim tamen etiam haec solutio adaptari potest ad casus, quibus $nn < 1$, ita ut sit

$$\partial s = \partial\varphi \sqrt{(1 - (1 - nn)\sin.\varphi^2)},$$

quae expressio posito $\sin.\varphi^2 = 1 - \cos.\varphi^2$ abit in hanc

$$\partial s = \partial\varphi \sqrt{(nn + (1 - nn)\cos.\varphi^2)},$$

et posito brevitatis gratia $1 - nn = kk$ fiet

$$\partial s = \partial\varphi \sqrt{(nn + kk\cos.\varphi^2)},$$

ubi notetur esse $nn + kk = 1$.

26. Huic ergo formulae statim satisfiet ponendo

$$\partial x = n\partial\varphi \quad \text{et} \quad \partial y = k\partial\varphi\cos.\varphi,$$

unde autem curva resultaret transcendens; quare ut curvas algebraicas eruamus, statuamus ut ante

$$\partial x = n\partial\varphi\sin.\lambda\varphi + k\partial\varphi\cos.\varphi\cos.\lambda\varphi$$

et

$$\partial y = n\partial\varphi\cos.\lambda\varphi - k\partial\varphi\cos.\varphi\sin.\lambda\varphi,$$

unde ambo valores prodibunt algebraici, dum ne sit $\lambda = \pm 1$.

27. Reductione igitur solita in usum vocata nanciscemur has formulas

$$\frac{2\,\partial x}{\partial \varphi} = 2n \sin.\lambda\varphi + k \cos.(\lambda+1)\varphi + k \cos.(\lambda-1)\varphi$$

et

$$\frac{2\,\partial y}{\partial \varphi} = 2n \cos.\lambda\varphi - k \sin.(\lambda+1)\varphi - k \sin.(\lambda-1)\varphi,$$

unde integrando deducimus

$$2x = -\frac{2n}{\lambda} \cos.\lambda\varphi + \frac{k}{\lambda+1} \sin.(\lambda+1)\varphi + \frac{k}{\lambda-1} \sin.(\lambda-1)\varphi$$

et

$$2y = +\frac{2n}{\lambda} \sin.\lambda\varphi + \frac{k}{\lambda+1} \cos.(\lambda+1)\varphi + \frac{k}{\lambda-1} \cos.(\lambda-1)\varphi,$$

quae curvae itidem maxime discrepant a praecedente solutione.

SCHOLION

28. Quamvis autem hae solutiones infinities infinitas suppeditent lineas curvas algebraicas problemati nostro satisfacientes, tamen vix affirmari posse videtur in his formulis omnes plane solutiones contineri; tam parum enim adhuc istud argumentum est elaboratum, ut vix quicquam certi in hoc negotio statui posse videatur, sed potius quaestio generalis, qua curvae algebraicae desiderantur, quarum longitudo per datam formulam integralem $\int V\partial v$ exprimatur, ubi V denotet functionem quamcunque ipsius v, tantopere etiamnunc tenebris obvoluta deprehenditur, ut solutionem paucissimis tantum casibus evolvere liceat, quemadmodum nobis solutio successit pro arcubus parabolicis et ellipticis. Si enim talis quaestio circa arcus hyperbolicos proponatur, fateri cogor nullo adhuc modo me vel unicam saltem curvam algebraicam eruere potuisse, cuius singuli arcus per formulam

$$\int \frac{\partial v}{vv} \sqrt{(1+v^4)}$$

Si enim v denotet abscissam hyperbolae aequilaterae inter- cata erit $y = \frac{1}{v}$ ideoque $\partial y = -\frac{\partial v}{vv}$, unde elementum arcus

$$\partial s = \frac{\partial v}{vv} \sqrt{(1+v^4)}.$$

Sin autem aequationem generalem pro hyperbola assumere velimus, qua est $y = n\sqrt{(1+vv)}$, elementum arcus inde nascitur

$$\partial s = \frac{\partial v \sqrt{(1+(nn+1)vv)}}{\sqrt{(1+vv)}},$$

quae formula ita comparata est, ut omnia artificia, quae quidem mihi detegere licuit, penitus frustretur. Quin etiam hic nullo modo calculus angulorum cum ullo successu in subsidium vocari potest. Neutiquam autem etiamnunc asseverare ausim praeter hyperbolam nullas alias dari curvas algebraicas, quarum longitudinem per arcus hyperbolicos metiri liceat, quemadmodum hoc de circulo audacter pronunciare non dubitavi. Hac igitur speculatione amplissimus campus aperitur, in quo geometrae non sine insigni fructu et Analyseos ulteriori perfectione elaborare poterunt.

DE CURVIS ALGEBRAICIS QUARUM LONGITUDO EXPRIMITUR HAC FORMULA INTEGRALI

$$\int \frac{v^{m-1}\partial v}{\sqrt{(1-v^{2n})}}$$

Convent. exhib. die 17 Iunii 1776

Commentatio 645 indicis ENESTROEMIANI

Nova acta academiae scientiarum Petropolitanae 6 (1788), 1790, p. 36—62

Summarium ibidem p. 84—85

SUMMARIUM

L'illustre Auteur de ce Mémoire, dont le génie se roidissoit constamment contre les difficultés, et qui dût à cette opiniâtreté, ou à cette force de caractère, la gloire d'en avoir surmonté tant dans sa vie, a fait un grand nombre de recherches sur les lignes courbes dont les arcs peuvent être exprimés par certaines formules intégrales. On se souviendra de ses recherches sur les courbes dont la longueur peut être mesurée par des arcs elliptiques et paraboliques[1]), desquelles nous avons rendu compte dans les extraits insérés dans l'Histoire de l'Académie pour l'année précédente. Toute cette matière est encore enveloppée de profondes ténèbres, et ce ne sera qu'à force de traiter beaucoup de cas particuliers, qu'on parviendra à y répandre de la lumière. C'est donc dans l'intention de contribuer quelque chose à l'approfondissement de ces mystères, que l'Auteur cherche ici des courbes algébriques dont la longueur de l'arc indéfini puisse être exprimée par la formule intégrale rapportée dans le titre du Mémoire. Il en trouve une quantité innombrable de tous les ordres, quoique la méthode qu'il employe n'épuise pas encore toutes les solutions possibles, comme M. EULER fait voir clairement par l'exemple des courbes rectifiables qui satisfont au problème. Au reste nous sommes obligés de renvoyer le Lecteur curieux de

1) L. EULERI Commentationes 638 et 639 (indicis ENESTROEMIANI); vide p. 151 et 163. A. K.

connoitre la méthode de l'Auteur, au Mémoire même; car il est impossible d'en donner une idée suffisante, sans répéter une bonne partie des calculs longs et pénibles, par lesquels il faut passer pour déterminer les coordonnées des courbes satisfaisantes.

Le Mémoire est terminé par la solution d'un problème beaucoup plus général, dans lequel on demande des courbes algébriques dont les arcs indéfinis pussent être exprimés par cette forme intégrale

$$\int \frac{v^{m-1}\partial v}{\sqrt{(1-v^{2n})}}(a + bv^{2n} + cv^{4n} + dv^{6n} + \text{etc.}).$$

1. Cum methodus certa huiusmodi problemata solvendi, quibus curvae algebraicae requiruntur, quarum longitudo per datam formulam integralem exprimatur, etiamnunc densissimis tenebris sit involuta, plurimum ad fines Analyseos amplificandos sine dubio conferet, si plura huius generis problemata particularia omni studio evolvantur, siquidem tum demum sperare licebit fore, ut tandem haec mysteria Analyseos ulterius penetremus. Hunc in finem constitui formulam propositam accuratius perscrutari, cuius quidem duo casus nulla prorsus laborant difficultate, alter scilicet, quando $m = 2n$ vel etiam $m = 4n$ vel $m = 6n$ etc., quia tum formula integrationem admittit ideoque omnes plane curvae algebraicae rectificabiles satisfacere sunt censendae, alter vero est $m = n$; tum enim nostra formula posito $v^n = z$ abit in hanc $\frac{\partial z}{n\sqrt{(1-zz)}}$ ideoque arcum circularem refert. Constat autem iam satis praeter circulum nullas alias lineas curvas algebraicas satisfacere posse[1]).

2. Ut autem nostram quaestionem in genere solvamus, designemus coordinatas curvarum quaesitarum litteris x et y, ipsos autem earum arcus littera s, ita ut sit $\partial s = \sqrt{(\partial x^2 + \partial y^2)}$; et quaestio huc redit, ut pro x et y eiusmodi functiones algebraicae quantitatis v investigentur, ut inde fiat

$$\partial s = \int \frac{v^{m-1}\partial v}{\sqrt{(1-v^{2n})}},$$

cui quidem quaestioni satisfieri posset, si eiusmodi angulos ω assignare liceret, ut ambae istae formulae

$$\partial x = \frac{v^{m-1}\partial v \cos.\omega}{\sqrt{(1-v^{2n})}} \quad \text{et} \quad \partial y = \frac{v^{m-1}\partial v \sin.\omega}{\sqrt{(1-v^{2n})}}$$

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1) Vide notam p. 83. A. K.

evaderent integrabiles. Verum nulla via patet in huiusmodi angulos inquirendi, nisi ipsa formula proposita ante in aliam formam ad calculum angulorum magis accommodatam transformetur.

3. Hunc in finem statuamus

$$v^n = \sin.\varphi,$$

ut fiat $\sqrt{(1-v^{2n})} = \cos.\varphi$; tum vero erit $v^m = \sin.\varphi^{\frac{m}{n}}$, ubi brevitatis gratia faciamus

$$\frac{m}{n} = \alpha + 1,$$

ut sit

$$v^m = \sin.\varphi^{\alpha+1},$$

unde differentiando erit

$$mv^{m-1}\partial v = (\alpha+1)\partial\varphi \cos.\varphi \sin.\varphi^\alpha,$$

ita ut nunc formula resolvenda proditura sit

$$\partial s = \frac{\alpha+1}{m}\partial\varphi \sin.\varphi^\alpha = \frac{1}{n}\partial\varphi \sin.\varphi^\alpha.$$

Quo autem hoc negotium facilius expediamus, duas sequentes formulas

$$z = \sin.\lambda\varphi \sin.\varphi^{\alpha+1} \quad \text{et} \quad z = \cos.\lambda\varphi \sin.\varphi^{\alpha+1}$$

studio evolvamus.

EVOLUTIO FORMULAE PRIORIS $z = \sin.\lambda\varphi \sin.\varphi^{\alpha+1}$

4. Quodsi istam formulam differentiemus, prodibit

$$\frac{\partial z}{\partial\varphi} = \lambda \cos.\lambda\varphi \sin.\varphi^{\alpha+1} + (\alpha+1)\sin.\lambda\varphi \cos.\varphi \sin.\varphi^\alpha$$

sive

$$\frac{\partial z}{\partial\varphi} = \sin.\varphi^\alpha(\lambda\cos.\lambda\varphi \sin.\varphi + (\alpha+1)\sin.\lambda\varphi\cos.\varphi).$$

Iam in subsidium vocentur reductiones notissimae

$$\sin.\lambda\varphi \cos.\varphi = \frac{1}{2}\sin.(\lambda+1)\varphi + \frac{1}{2}\sin.(\lambda-1)\varphi$$

et

$$\cos.\lambda\varphi \sin.\varphi = \frac{1}{2}\sin.(\lambda+1)\varphi - \frac{1}{2}\sin.(\lambda-1)\varphi,$$

quibus valoribus substitutis reperiemus

$$\frac{2\partial z}{\partial \varphi} = \sin.\varphi^{\alpha}[(\alpha+1+\lambda)\sin.(\lambda+1)\varphi + (\alpha+1-\lambda)\sin.(\lambda-1)\varphi],$$

unde colligimus hanc integrationem

$$2\sin.\lambda\varphi\sin.\varphi^{\alpha+1} = (\alpha+1+\lambda)\int\partial\varphi\sin.\varphi^{\alpha}\sin.(\lambda+1)\varphi$$
$$+(\alpha+1-\lambda)\int\partial\varphi\sin.\varphi^{\alpha}\sin.(\lambda-1)\varphi,$$

ubi notetur esse $\partial\varphi\sin.\varphi^{\alpha} = n\partial s$.

5. Ponamus nunc statim

$$\lambda = \alpha + 1 = \frac{m}{n}$$

atque integratio inventa praebebit

$$\sin.\frac{m}{n}\varphi\sin.\varphi^{\frac{m}{n}} = m\int\partial s\sin.\frac{m+n}{n}\varphi,$$

unde vicissim conficitur

$$\int\partial s\sin.\frac{m+n}{n}\varphi = \frac{1}{m}\sin.\frac{m}{n}\varphi\sin.\varphi^{\frac{m}{n}}.$$

Hinc, si fuerit $\partial y = \partial s\sin.\frac{m+n}{n}\varphi$, valor ipsius y erit algebraicus.

6. Sumamus nunc in nostra integratione generali

$$\lambda = 1 + \frac{m+n}{n} = \frac{m+2n}{n}$$

atque habebimus

$$\sin.\frac{m+2n}{n}\varphi\sin.\varphi^{\frac{m}{n}} = (m+n)\int\partial s\sin.\frac{m+3n}{n}\varphi - n\int\partial s\sin.\frac{m+n}{n}\varphi,$$

ubi valorem integralis posterioris iam ante definivimus, quare integrale prius sequenti modo exprimetur

$$\int\partial s\sin.\frac{m+3n}{n}\varphi = \frac{1}{m+n}\sin.\frac{m+2n}{n}\varphi\sin.\varphi^{\frac{m}{n}} + \frac{n}{m(m+n)}\sin.\frac{m}{n}\varphi\sin.\varphi^{\frac{m}{n}}$$

sive

$$\int\partial s\sin.\frac{m+3n}{n}\varphi = \frac{1}{m+n}\sin.\varphi^{\frac{m}{n}}\left(\sin.\frac{m+2n}{n}\varphi + \frac{n}{m}\sin.\frac{m}{n}\varphi\right).$$

7. Ponamus porro in forma generali

$$\lambda - 1 = \frac{m+3n}{n} \quad \text{sive} \quad \lambda = \frac{m+4n}{n}$$

ac reperiemus

$$\sin.\frac{m+4n}{n}\varphi \sin.\varphi^{\frac{m}{n}} = (m+2n)\int \partial s \sin.\frac{m+5n}{n}\varphi - 2n\int \partial s \sin.\frac{m+3n}{n}\varphi,$$

ubi cum posterius integrale modo invenerimus, prius sequenti modo exprimetur

$$\int \partial s \sin.\frac{m+5n}{n}\varphi = \frac{1}{m+2n}\sin.\frac{m+4n}{n}\varphi \sin.\varphi^{\frac{m}{n}} + \frac{2n}{m+2n}\int \partial s \sin.\frac{m+3n}{n}\varphi.$$

8. Simili modo statuamus nunc

$$\lambda - 1 = \frac{m+5n}{n} \quad \text{sive} \quad \lambda = \frac{m+6n}{n}$$

atque nanciscimur sequentem integrationem

$$\sin.\frac{m+6n}{n}\varphi \sin.\varphi^{\frac{m}{n}} = (m+3n)\int \partial s \sin.\frac{m+7n}{n}\varphi - 3n\int \partial s \sin.\frac{m+5n}{n}\varphi,$$

unde concludimus fore

$$\int \partial s \sin.\frac{m+7n}{n}\varphi = \frac{1}{m+3n}\sin.\frac{m+6n}{n}\varphi \sin.\varphi^{\frac{m}{n}} + \frac{3n}{m+3n}\int \partial s \sin.\frac{m+5n}{n}\varphi.$$

9. Lex, qua hae formulae continuo ulterius procedunt, satis est manifesta, ita ut non opus sit calculum ultra prosequi. At quo eas distinctius obtutui exponamus, sit brevitatis gratia $\sin.\varphi^{\frac{m}{n}} = \Phi$ et singulae formulae integrales hinc oriundae ita se habebunt

I. $\int \partial s \sin.\frac{m+n}{n}\varphi = \frac{1}{m}\Phi \sin.\frac{m}{n}\varphi$

II. $\int \partial s \sin.\frac{m+3n}{n}\varphi = \frac{1}{m+n}\Phi \sin.\frac{m+2n}{n}\varphi + \frac{n}{m+n}\int \partial s \sin.\frac{m+n}{n}\varphi$

III. $\int \partial s \sin.\frac{m+5n}{n}\varphi = \frac{1}{m+2n}\Phi \sin.\frac{m+4n}{n}\varphi + \frac{2n}{m+2n}\int \partial s \sin.\frac{m+3n}{n}\varphi$

$$\text{IV. } \int \partial s \sin. \frac{m+7n}{n}\varphi = \frac{1}{m+3n}\Phi \sin. \frac{m+6n}{n}\varphi + \frac{3n}{m+3n}\int \partial s \sin. \frac{m+5n}{n}\varphi$$

$$\text{V. } \int \partial s \sin. \frac{m+9n}{n}\varphi = \frac{1}{m+4n}\Phi \sin. \frac{m+8n}{n}\varphi + \frac{4n}{m+4n}\int \partial s \sin. \frac{m+7n}{n}\varphi$$

$$\text{VI. } \int \partial s \sin. \frac{m+11n}{n}\varphi = \frac{1}{m+5n}\Phi \sin. \frac{m+10n}{n}\varphi + \frac{5n}{m+5n}\int \partial s \sin. \frac{m+9n}{n}\varphi$$

etc. etc.

10. Quodsi iam in singulis his formulis valores integralis praecedentis substituamus, adipiscemur sequentes integrationes ad nostrum usum accommodatas

$$\text{I. } \int \partial s \sin. \frac{m+n}{n}\varphi = \frac{\Phi}{m}\sin. \frac{m}{n}\varphi$$

$$\text{II. } \int \partial s \sin. \frac{m+3n}{n}\varphi = \frac{\Phi}{m+n}\left(\sin. \frac{m+2n}{n}\varphi + \frac{n}{m}\sin. \frac{m}{n}\varphi\right)$$

$$\text{III. } \int \partial s \sin. \frac{m+5n}{n}\varphi = \frac{\Phi}{m+2n}\left\{\begin{aligned} &\sin. \frac{m+4n}{n}\varphi + \frac{2n}{m+n}\sin. \frac{m+2n}{n}\varphi \\ &+ \frac{n\cdot 2n}{m(m+n)}\sin. \frac{m}{n}\varphi \end{aligned}\right\}$$

$$\text{IV. } \int \partial s \sin. \frac{m+7n}{n}\varphi = \frac{\Phi}{m+3n}\left\{\begin{aligned} &\sin. \frac{m+6n}{n}\varphi + \frac{3n}{m+2n}\sin. \frac{m+4n}{n}\varphi \\ &+ \frac{2n\cdot 3n}{(m+n)(m+2n)}\sin. \frac{m+2n}{n}\varphi \\ &+ \frac{n\cdot 2n\cdot 3n}{m(m+n)(m+2n)}\sin. \frac{m}{n}\varphi \end{aligned}\right\}$$

$$\text{V. } \int \partial s \sin. \frac{m+9n}{n}\varphi = \frac{\Phi}{m+4n}\left\{\begin{aligned} &\sin. \frac{m+8n}{n}\varphi + \frac{4n}{m+3n}\sin. \frac{m+6n}{n}\varphi \\ &+ \frac{3n\cdot 4n}{(m+2n)(m+3n)}\sin. \frac{m+4n}{n}\varphi \\ &+ \frac{2n\cdot 3n\cdot 4n}{(m+n)(m+2n)(m+3n)}\sin. \frac{m+2n}{n}\varphi \\ &+ \frac{n\cdot 2n\cdot 3n\cdot 4n}{m(m+n)(m+2n)(m+3n)}\sin. \frac{m}{n}\varphi \end{aligned}\right\}$$

$$\text{VI. } \int \partial s \sin. \frac{m+11n}{n}\varphi = \frac{\Phi}{m+5n} \left\{ \begin{aligned} & \sin. \frac{m+10n}{n}\varphi + \frac{5n}{m+4n}\sin. \frac{m+8n}{n}\varphi \\ & + \frac{4n\cdot 5n}{(m+3n)(m+4n)}\sin. \frac{m+6n}{n}\varphi \\ & + \frac{3n\cdot 4n\cdot 5n}{(m+2n)(m+3n)(m+4n)}\sin. \frac{m+4n}{n}\varphi \\ & + \frac{2n\cdot 3n\cdot 4n\cdot 5n}{(m+n)(m+2n)(m+3n)(m+4n)}\sin. \frac{m+2n}{n}\varphi \\ & + \frac{n\cdot 2n\cdot 3n\cdot 4n\cdot 5n}{m(m+n)(m+2n)(m+3n)(m+4n)}\sin. \frac{m}{n}\varphi \end{aligned} \right\}$$

etc. etc.

ubi tantum meminisse oportet esse $\Phi = \sin.\varphi^{\frac{m}{n}}$.

11. Hae formulae adhuc concinniores reddi possunt ponendo $\frac{m}{n} = k$, ut sit $\Phi = \sin.\varphi^k$; tum vero sequentes orientur formulae integrales

$$\text{I. } \int \partial s \sin.(k+1)\varphi = \frac{\sin.\varphi^k}{nk}\sin.k\varphi$$

$$\text{II. } \int \partial s \sin.(k+3)\varphi = \frac{\sin.\varphi^k}{n(k+1)}\left(\sin.(k+2)\varphi + \frac{1}{k}\sin.k\varphi\right)$$

$$\text{III. } \int \partial s \sin.(k+5)\varphi = \frac{\sin.\varphi^k}{n(k+2)} \left\{ \begin{aligned} & \sin.(k+4)\varphi + \frac{2}{k+1}\sin.(k+2)\varphi \\ & \qquad + \frac{1\cdot 2}{k(k+1)}\sin.k\varphi \end{aligned} \right\}$$

$$\text{IV. } \int \partial s \sin.(k+7)\varphi = \frac{\sin.\varphi^k}{n(k+3)} \left\{ \begin{aligned} & \sin.(k+6)\varphi + \frac{3}{k+2}\sin.(k+4)\varphi \\ & + \frac{2\cdot 3}{(k+1)(k+2)}\sin.(k+2)\varphi \\ & + \frac{1\cdot 2\cdot 3}{k(k+1)(k+2)}\sin.k\varphi \end{aligned} \right\}.$$

$$\text{V. } \int \partial s \sin.(k+9)\varphi = \frac{\sin.\varphi^k}{n(k+4)} \left\{ \begin{aligned} &\sin.(k+8)\varphi + \frac{4}{k+3}\sin.(k+6)\varphi \\ &+ \frac{3\cdot 4}{(k+2)(k+3)}\sin.(k+4)\varphi \\ &+ \frac{2\cdot 3\cdot 4}{(k+1)(k+2)(k+3)}\sin.(k+2)\varphi \\ &+ \frac{1\cdot 2\cdot 3\cdot 4}{k(k+1)(k+2)(k+3)}\sin.k\varphi \end{aligned} \right\}$$

$$\text{VI. } \int \partial s \sin.(k+11)\varphi = \frac{\sin.\varphi^k}{n(k+5)} \left\{ \begin{aligned} &\sin.(k+10)\varphi + \frac{5}{k+4}\sin.(k+8)\varphi \\ &+ \frac{4\cdot 5}{(k+3)(k+4)}\sin.(k+6)\varphi \\ &+ \frac{3\cdot 4\cdot 5}{(k+2)(k+3)(k+4)}\sin.(k+4)\varphi \\ &+ \frac{2\cdot 3\cdot 4\cdot 5}{(k+1)(k+2)(k+3)(k+4)}\sin.(k+2)\varphi \\ &+ \frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{k(k+1)(k+2)(k+3)(k+4)}\sin.k\varphi \end{aligned} \right\}$$

12. Hinc igitur patet, si i denotet numerum positivum quemcunque, generatim integrale huius formae $\int \partial s \sin.(k+2i+1)\varphi$ actu exhiberi posse; lege enim progressionis probe observata erit

$$\int \partial s \sin.(k+2i+1)\varphi$$

$$= \frac{\sin.\varphi^k}{n(k+i)} \left\{ \begin{aligned} &\sin.(k+2i)\varphi + \frac{i}{k+i-1}\sin.(k+2i-2)\varphi \\ &+ \frac{(i-1)i}{(k+i-2)(k+i-1)}\sin.(k+2i-4)\varphi \\ &+ \frac{(i-2)(i-1)i}{(k+i-3)(k+i-2)(k+i-1)}\sin.(k+2i-6)\varphi + \text{etc.} \end{aligned} \right\}$$

Ubi tantum observetur haec integralia quandoque incongrua fieri posse, quod evenit, quoties in denominatoribus harum fractionum factor quispiam nihilo fit aequalis, siquidem his casibus integrale non amplius erit algebraicum. Hoc autem contingere poterit, quoties k, hoc est $\frac{m}{n}$, fuerit vel $=0$ vel

numerus integer negativus ipsi i aequalis vel minor; sin autem iste valor negativus ipsius k superet i, memoratum incommodum non amplius erit metuendum.

EVOLUTIO FORMULAE POSTERIORIS $z = \cos.\lambda\varphi \sin.\varphi^{\alpha+1}$

13. Quodsi haec formula differentietur, prodibit

$$\frac{\partial z}{\partial \varphi} = \sin.\varphi^{\alpha}((\alpha+1)\cos.\varphi\cos.\lambda\varphi - \lambda\sin.\varphi\sin.\lambda\varphi).$$

Cum nunc per notas reductiones sit

$$\cos.\varphi\cos.\lambda\varphi = \frac{1}{2}\cos.(\lambda-1)\varphi + \frac{1}{2}\cos.(\lambda+1)\varphi$$

et

$$\sin.\varphi\sin.\lambda\varphi = \frac{1}{2}\cos.(\lambda-1)\varphi - \frac{1}{2}\cos.(\lambda+1)\varphi,$$

his substitutis pervenietur ad hanc formam

$$\frac{2\partial z}{\partial \varphi} = \sin.\varphi^{\alpha}((\alpha+1-\lambda)\cos.(\lambda-1)\varphi + (\alpha+1+\lambda)\cos.(\lambda+1)\varphi),$$

unde deducitur ista integratio

$$2\cos.\lambda\varphi\sin.\varphi^{\alpha+1} = (\alpha+1-\lambda)\int\partial\varphi\cos.(\lambda-1)\varphi\sin.\varphi^{\alpha} + (\alpha+1+\lambda)\int\partial\varphi\cos.(\lambda+1)\varphi\sin.\varphi^{\alpha}.$$

14. Quoniam igitur supra vidimus esse

$$\partial\varphi\sin.\varphi^{\alpha} = n\,\partial s,$$

ob $\alpha+1 = \frac{m}{n} = k$ ista integratio ad hanc formam redibit

$$2\cos.\lambda\varphi\sin.\varphi^{k} = n(k-\lambda)\int\partial s\cos.(\lambda-1)\varphi + n(k+\lambda)\int\partial s\cos.(\lambda+1)\varphi,$$

ex qua deducimus

$$\int\partial s\cos.(\lambda+1)\varphi = \frac{2}{n(k+\lambda)}\cos.\lambda\varphi\sin.\varphi^{k} - \frac{k-\lambda}{k+\lambda}\int\partial s\cos.(\lambda-1)\varphi.$$

15. Ex hac forma generali iam derivemus casus speciales, ut supra fecimus, ac primo quidem sumamus $\lambda = k$, ut obtineamus istud quasi principium sequentium integrationum, scilicet

$$\text{I. } \int \partial s \cos.(k+1)\varphi = \frac{\sin.\varphi^k}{nk}\cos.k\varphi.$$

Sumamus nunc $\lambda - 1 = k + 1$ sive $\lambda = k + 2$ et integratio generalis dabit

$$\text{II. } \int \partial s \cos.(k+3)\varphi = \frac{\sin.\varphi^k}{n(k+1)}\cos.(k+2)\varphi + \frac{1}{k+1}\int \partial s \cos.(k+1)\varphi.$$

Fiat nunc $\lambda - 1 = k + 3$ sive $\lambda = k + 4$ ac prodibit

$$\text{III. } \int \partial s \cos.(k+5)\varphi = \frac{\sin.\varphi^k}{n(k+2)}\cos.(k+4)\varphi + \frac{2}{k+2}\int \partial s \cos.(k+3)\varphi.$$

Sit iam ulterius $\lambda - 1 = k + 5$ sive $\lambda = k + 6$ ac prodibit

$$\text{IV. } \int \partial s \cos.(k+7)\varphi = \frac{\sin.\varphi^k}{n(k+3)}\cos.(k+6)\varphi + \frac{3}{k+3}\int \partial s \cos.(k+5)\varphi.$$

Sit porro $\lambda - 1 = k + 7$ sive $\lambda = k + 8$ ac fiet

$$\text{V. } \int \partial s \cos.(k+9)\varphi = \frac{\sin.\varphi^k}{n(k+4)}\cos.(k+8)\varphi + \frac{4}{k+4}\int \partial s \cos.(k+7)\varphi$$

etc. etc.

16. Quodsi iam in singulis formulis integralia praecedentia substituamus, nanciscemur sequentes integrationes

$$\text{I. } \int \partial s \cos.(k+1)\varphi = \frac{\sin.\varphi^k}{nk}\cos.k\varphi$$

$$\text{II. } \int \partial s \cos.(k+3)\varphi = \frac{\sin.\varphi^k}{n(k+1)}\left(\cos.(k+2)\varphi + \frac{1}{k}\cos.k\varphi\right)$$

$$\text{III. } \int \partial s \cos.(k+5)\varphi = \frac{\sin.\varphi^k}{n(k+2)}\left\{\begin{aligned}&\cos.(k+4)\varphi + \frac{2}{k+1}\cos.(k+2)\varphi\\ &\qquad + \frac{1\cdot 2}{k(k+1)}\cos.k\varphi\end{aligned}\right\}$$

$$\text{IV. } \int \partial s \cos.(k+7)\varphi = \frac{\sin.\varphi^k}{n(k+3)}\left\{\begin{aligned}&\cos.(k+6)\varphi + \frac{3}{k+2}\cos.(k+4)\varphi\\ &\quad + \frac{2\cdot 3}{(k+1)(k+2)}\cos.(k+2)\varphi\\ &\quad + \frac{1\cdot 2\cdot 3}{k(k+1)(k+2)}\cos.k\varphi\end{aligned}\right\}$$

$$\text{V. } \int \partial s \cos.(k+9)\varphi = \frac{\sin.\varphi^k}{n(k+4)} \left\{ \begin{array}{l} \cos.(k+8)\varphi + \frac{4}{k+3}\cos.(k+6)\varphi \\ + \frac{3\cdot 4}{(k+2)(k+3)}\cos.(k+4)\varphi \\ + \frac{2\cdot 3\cdot 4}{(k+1)(k+2)(k+3)}\cos.(k+2)\varphi \\ + \frac{1\cdot 2\cdot 3\cdot 4}{k(k+1)(k+2)(k+3)}\cos.k\varphi \end{array} \right\}$$

$$\text{VI. } \int \partial s \cos.(k+11)\varphi = \frac{\sin.\varphi^k}{n(k+5)} \left\{ \begin{array}{l} \cos.(k+10)\varphi + \frac{5}{k+4}\cos.(k+8)\varphi \\ + \frac{4\cdot 5}{(k+3)(k+4)}\cos.(k+6)\varphi \\ + \frac{3\cdot 4\cdot 5}{(k+2)(k+3)(k+4)}\cos.(k+4)\varphi \\ + \frac{2\cdot 3\cdot 4\cdot 5}{(k+1)(k+2)(k+3)(k+4)}\cos.(k+2)\varphi \\ + \frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{k(k+1)(k+2)(k+3)(k+4)}\cos.k\varphi \end{array} \right\}$$

etc. etc.

quae formulae a praecedentibus hoc tantum discrepant, ut sinus angulorum hic in cosinus sint transmutati.

17. Ex his igitur casibus facile deducimus sequentem formulam integralem

$$\int \partial s \cos.(k+2i+1)\varphi$$

$$= \frac{\sin.\varphi^k}{n(k+i)} \left\{ \begin{array}{l} \cos.(k+2i)\varphi + \frac{i}{k+i-1}\cos.(k+2i-2)\varphi \\ + \frac{(i-1)i}{(k+i-2)(k+i-1)}\cos.(k+2i-4)\varphi \\ + \frac{(i-2)(i-1)i}{(k+i-3)(k+i-2)(k+i-1)}\cos.(k+2i-6)\varphi + \text{etc.} \end{array} \right\}$$

His igitur duabus formulis generalibus evolutis quaestionem propositam sequenti modo facile resolvere licebit.

PROBLEMA

18. *Invenire curvas algebraicas, quarum longitudo ita exprimatur, ut eius rcus quicunque indefinitus sit*

$$s = \int \frac{v^{m-1}\partial v}{\sqrt{(1-v^{2n})}}.$$

SOLUTIO

Quaeratur primo angulus φ, ut sit

$$\sin.\varphi = v^n \quad \text{ideoque} \quad \cos.\varphi = \sqrt{(1-v^{2n})};$$

um vero posito brevitatis gratia $\frac{m}{n} = k$ fiet

$$\partial s = \frac{1}{n}\partial\varphi \sin.\varphi^{k-1};$$

uod cum sit elementum curvae, si coordinatae orthogonales vocentur x et y, n genere habebimus

$$\partial x = \partial s \cos.\omega \quad \text{et} \quad \partial y = \partial s \sin.\omega,$$

uandoquidem hinc prodit $\partial x^2 + \partial y^2 = \partial s^2$.

19. Totum negotium ergo huc redit, cuiusmodi angulos pro ω accipi porteat, ut binae istae formulae differentiales evadant integrabiles, id quod stendimus semper fieri sumendo $\omega = (k+2i+1)\varphi$, ita ut sit

$$x = \int \partial s \cos.(k+2i+1)\varphi \quad \text{et} \quad y = \int \partial s \sin.(k+2i+1)\varphi;$$

um enim habebimus sequentes formulas algebraicas

$$x = \frac{\sin.\varphi^k}{n(k+i)}\left\{\begin{array}{c} \cos.(k+2i)\varphi + \frac{i}{k+i-1}\cos.(k+2i-2)\varphi \\ + \frac{i(i-1)}{(k+i-1)(k+i-2)}\cos.(k+2i-4)\varphi \\ + \frac{i(i-1)(i-2)}{(k+i-1)(k+i-2)(k+i-3)}\cos.(k+2i-6)\varphi \\ + \frac{i(i-1)(i-2)(i-3)}{(k+i-1)(k+i-2)(k+i-3)(k+i-4)}\cos.(k+2i-8)\varphi \\ + \text{etc.} \end{array}\right\}$$

et

$$y = \frac{\sin.\varphi^k}{n(k+i)} \left\{ \begin{array}{c} \sin.(k+2i)\varphi + \frac{i}{k+i-1} \sin.(k+2i-2)\varphi \\ + \frac{i(i-1)}{(k+i-1)(k+i-2)} \sin.(k+2i-4)\varphi \\ + \frac{i(i-1)(i-2)}{(k+i-1)(k+i-2)(k+i-3)} \sin.(k+2i-6)\varphi \\ + \frac{i(i-1)(i-2)(i-3)}{(k+i-1)(k+i-2)(k+i-3)(k+i-4)} \sin.(k+2i-8)\varphi \\ + \text{etc.} \end{array} \right\}$$

ubi loco i omnes numeros integros positivos a 0 in infinitum usque accipere licet; unde sequentes solutiones speciales evolvisse iuvabit.

I. SOLUTIO SPECIALIS QUA $i = 0$

20. Hinc igitur resultabit solutio simplicissima; ambae enim coordinatae x et y ita exprimentur, ut sit

$$x = \frac{\sin.\varphi^k \cos.k\varphi}{nk} \quad \text{et} \quad y = \frac{\sin.\varphi^k \sin.k\varphi}{nk},$$

quae solutio semper est realis, nisi fuerit $k = 0$; tum autem foret quoque $m = 0$ et

$$\partial s = \frac{\partial v}{v\sqrt{(1 - v^{2n})}} = \frac{1}{n} \frac{\partial \varphi}{\sin.\varphi},$$

unde fit

$$s = \frac{1}{n} l \text{ tang.} \frac{1}{2}\varphi,$$

sicque arcus per simplicem logarithmum exprimeretur; tales autem curvas algebraicas nullo modo exhiberi posse satis est evictum. Caeterum pro omnibus reliquis casibus, quemcunque valorem rationalem habuerit k, semper erit

$$xx + yy = \frac{\sin.\varphi^{2k}}{nnkk} = \frac{v^{2nk}}{nnkk} = \frac{v^{2m}}{mm}$$

ideoque chorda

$$\sqrt{(xx + yy)} = \frac{v^m}{m}.$$

II. SOLUTIO SPECIALIS QUA $i=1$

21. Hoc igitur casu ambae coordinatae ita erunt expressae

$$x = \frac{\sin.\varphi^k}{n(k+1)}\left(\cos.(k+2)\varphi + \frac{1}{k}\cos.k\varphi\right)$$

et

$$y = \frac{\sin.\varphi^k}{n(k+1)}\left(\sin.(k+2)\varphi + \frac{1}{k}\sin.k\varphi\right),$$

unde conficitur chorda

$$\sqrt{(xx+yy)} = \frac{\sin.\varphi^k}{n(k+1)}\sqrt{\left(1+\frac{1}{kk}+\frac{2}{k}\cos.2\varphi\right)}$$

sive

$$\sqrt{(xx+yy)} = \frac{v^m}{m(m+n)}\sqrt{((m+n)^2 - 4mnv^{2n})},$$

haecque solutio semper valebit praeter duos casus excipiendos, qui sunt vel $k=0$ vel $k=-1$.

III. SOLUTIO SPECIALIS QUA $i=2$

22. Hoc igitur casu ambae coordinatae erunt ita expressae

$$x = \frac{\sin.\varphi^k}{n(k+2)}\left(\cos.(k+4)\varphi + \frac{2}{k+1}\cos.(k+2)\varphi + \frac{2}{k+1}\cdot\frac{1}{k}\cos.k\varphi\right)$$

et

$$y = \frac{\sin.\varphi^k}{n(k+2)}\left(\sin.(k+4)\varphi + \frac{2}{k+1}\sin.(k+2)\varphi + \frac{2}{k+1}\cdot\frac{1}{k}\sin.k\varphi\right).$$

Hic igitur tres casus excipi oportet, quibus hae formulae cessant esse algebraicae, primo scilicet si $k=0$, secundo si $k=-1$, tertio si $k=-2$.

IV. SOLUTIO SPECIALIS QUA $i=3$

23. Hoc igitur casu ambae coordinatae sequenti modo reperientur expressae

$$x = \frac{\sin.\varphi^k}{n(k+3)}\left\{\begin{array}{c}\cos.(k+6)\varphi + \frac{3}{k+2}\cos.(k+4)\varphi \\ + \frac{3}{k+2}\cdot\frac{2}{k+1}\cos.(k+2)\varphi + \frac{3}{k+2}\cdot\frac{2}{k+1}\cdot\frac{1}{k}\cos.k\varphi\end{array}\right\}$$

$$y = \frac{\sin.\varphi^k}{n(k+3)} \left\{ \begin{array}{c} \sin.(k+6)\varphi + \frac{3}{k+2}\sin.(k+4)\varphi \\ + \frac{3}{k+2}\cdot\frac{2}{k+1}\sin.(k+2)\varphi + \frac{3}{k+2}\cdot\frac{2}{k+1}\cdot\frac{1}{k}\sin.k\varphi \end{array} \right\}$$

Hae ergo formulae quatuor casibus erunt inutiles

$$1.\ k=0, \quad 2.\ k=-1, \quad 3.\ k=-2, \quad 4.\ k=-3,$$

quippe quibus termini in infinitum excrescentes abirent in arcus circulares neque igitur formulae amplius essent algebraicae.

V. SOLUTIO SPECIALIS QUA $i = 4$.

24. Hoc igitur casu coordinatae sequenti modo exprimentur

$$x = \frac{\sin.\varphi^k}{n(k+4)} \left\{ \begin{array}{l} \cos.(k+8)\varphi + \frac{4}{k+3}\cos.(k+6)\varphi \\ + \frac{4}{k+3}\cdot\frac{3}{k+2}\cos.(k+4)\varphi \\ + \frac{4}{k+3}\cdot\frac{3}{k+2}\cdot\frac{2}{k+1}\cos.(k+2)\varphi \\ + \frac{4}{k+3}\cdot\frac{3}{k+2}\cdot\frac{2}{k+1}\cdot\frac{1}{k}\cos.k\varphi \end{array} \right\}$$

$$y = \frac{\sin.\varphi^k}{n(k+4)} \left\{ \begin{array}{l} \sin.(k+8)\varphi + \frac{4}{k+3}\sin.(k+6)\varphi \\ + \frac{4}{k+3}\cdot\frac{3}{k+2}\sin.(k+4)\varphi \\ + \frac{4}{k+3}\cdot\frac{3}{k+2}\cdot\frac{2}{k+1}\sin.(k+2)\varphi \\ + \frac{4}{k+3}\cdot\frac{3}{k+2}\cdot\frac{2}{k+1}\cdot\frac{1}{k}\sin k\varphi \end{array} \right\}$$

ubi manifestum est has formulas praeter quatuor casus ante notatos insuper casu $k = -4$ fieri inutiles.

COROLLARIUM

25. Exceptis igitur casibus, quibus k aequatur numero integro negativo, methodus nostra semper suppeditat innumerabiles curvas algebraicas; neque tamen idcirco haec solutio pro generali est habenda, cum etiam casibus me-

moratis, quibus numerus solutionum nostrarum limitatur, nihilominus infinitas solutiones aliis methodis assignare liceat; ubi quidem semper excludi oportet casum $k=0$, quippe quo certum est nullas curvas algebraicas satisfacere posse. Innumerabilitatem solutionum pro casu $k=-1$ ostendisse operae erit pretium.

EVOLUTIO CASUS QUO $k=-1$

26. Hoc igitur casu nostra methodus unicam praebet curvam algebraicam his coordinatis contentam

$$x=-\frac{1}{n}\cdot\frac{\cos.\varphi}{\sin.\varphi} \quad \text{et} \quad y=\frac{1}{n},$$

quae ergo est linea recta axi parallela. Cum autem sit $\partial s=\frac{\partial\varphi}{n\sin.\varphi^2}$, erit

$$s=-\frac{1}{n}\cot.\varphi$$

sicque omnes plane curvae algebraicae rectificabiles hoc casu satisfaciunt. Sumta enim quacunque tali curva, cuius arcus s per formulam algebraicam exprimatur, semper assignari poterit angulus φ, ut fiat $-\frac{1}{n}\cot.\varphi=s$; unde patet praeter lineam rectam, quam invenimus, omnes plane curvas rectificabiles satisfacere.

COROLLARIUM

27. Cum igitur formula nostra differentialis

$$\partial s=\frac{v^{m-1}\partial v}{\sqrt{(1-v^{2n})}}$$

semper absolute evadat integrabilis, quoties k sive $\frac{m}{n}$ fuerit vel numerus integer positivus par vel etiam numerus integer negativus impar, manifestum est his omnibus casibus omnes plane curvas algebraicas rectificabiles perinde esse satisfacturas ideoque revera his casibus infinities plures curvae algebraicae nostro problemati satisfacient, quam nostra methodus nobis suppeditavit. Verum etiam, dummodo k sit numerus negativus integer par, semper innumerabiles curvas algebraicas assignare licet, quod pro casu $k=-2$ ostendisse sufficiet.

EVOLUTIO CASUS QUO $k = -2$

28. Hoc igitur casu methodus superior duas tantum nobis largitur curvas algebraicas, scilicet

$$1. \quad x = -\frac{\cos. 2\varphi}{2n \sin. \varphi^2} \quad \text{et} \quad y = \frac{\sin. 2\varphi}{2n \sin. \varphi^2}$$

$$2. \quad x = -\frac{1}{n \sin. \varphi^2}\left(1 - \frac{1}{2}\cos. 2\varphi\right) \quad \text{et} \quad y = +\frac{\sin. 2\varphi}{2n \sin. \varphi^2}.$$

Cum autem hoc casu sit $\partial s = \frac{\partial \varphi}{n \sin. \varphi^3}$, statuatur $\cot. \varphi = t$ eritque

$$\frac{\partial \varphi}{\sin. \varphi^2} = -\partial t \quad \text{ideoque} \quad \partial s = -\frac{\partial t}{n \sin. \varphi};$$

quia vero est $\sin. \varphi = \frac{1}{\sqrt{(1+tt)}}$, fiet

$$\partial s = -\frac{1}{n}\partial t \sqrt{(1+tt)};$$

quod cum sit elementum arcus parabolici, nuper[1]) iam demonstravi infinitas curvas algebraicas satisfacere atque hoc idem quoque tenendum est, si littera k cuicunque numero pari negativo maiori aequetur.

SCHOLION

29. Ex his iam facile colligere licet etiam in genere pro omnibus valoribus ipsius k revera infinities plures curvas algebraicas esse satisfacturas, quam methodus nostra nobis suppeditat, etiamsi adeo innumerabiles exhibeat. Interim tamen duos casus excipi necesse est, alterum, quo $k = 0$, pro quo iam notavimus nullas plane curvas algebraicas satisfacere; alterum vero, quo $k = 1$; cum enim sit $\partial s = \frac{1}{n}\partial \varphi$, arcus s ipse arcui circulari aequari deberet, cui conditioni solus circulus satisfacere est monstratus, id quod etiam nostrae solutiones manifesto declarabunt.

EVOLUTIO CASUS QUO $k = 1$

Pro hoc ergo casu solutio specialis prima praebet has coordinatas

$$x = \frac{1}{n} \sin. \varphi \cos. \varphi \quad \text{et} \quad y = \frac{1}{n} \sin. \varphi^2.$$

1) Commentatio 638 (indicis Enestroemiani); vide p. 151. A. K.

Cum igitur sit

$$x = \frac{1}{2n}\sin.2\varphi \quad \text{et} \quad y = \frac{1}{2n}(1-\cos.2\varphi),$$

erit $\frac{1}{2n}\cos.2\varphi = \frac{1}{2n} - y$; additis ergo quadratis erit

$$xx + \left(\frac{1}{2n} - y\right)^2 = \frac{1}{4nn},$$

quae aequatio manifesto est pro circulo.

31. Secunda vero solutio specialis pro hoc casu nobis dat

$$x = \frac{\sin.\varphi}{2n}(\cos.3\varphi + \cos.\varphi)$$

et

$$y = \frac{\sin.\varphi}{2n}(\sin.3\varphi + \sin.\varphi),$$

quae formulae per reductiones notas abeunt in has

$$4nx = \sin.4\varphi \quad \text{et} \quad 4ny = 1 - \cos.4\varphi$$

ideoque $\cos.4\varphi = 1 - 4ny$. Additis igitur quadratis orietur

$$16nnxx + (1-4ny)^2 = 1,$$

quae itidem est pro circulo.

32. Simili modo solutio specialis tertia praebet

$$x = \frac{\sin.\varphi}{3n}(\cos.5\varphi + \cos.3\varphi + \cos.\varphi)$$

et

$$y = \frac{\sin.\varphi}{3n}(\sin.5\varphi + \sin.3\varphi + \sin.\varphi),$$

quae pariter more solito reductae dant

$$6nx = \sin.6\varphi \quad \text{et} \quad 6ny = 1 - \cos.6\varphi,$$

unde si angulum 6φ eliminemus, manifesto resultat aequatio ad circulum.

33. Quin etiam hoc idem in genere ostendere licet, quandoquidem sumto $k = 1$ reperitur

$$x = \frac{\sin.\varphi}{n(i+1)}\left\{\begin{array}{c}\cos.(2i+1)\varphi + \cos.(2i-1)\varphi\\ + \cos.(2i-3)\varphi + \cos.(2i-5)\varphi\\ + \text{etc.}\ \ldots + \cos.\varphi\end{array}\right\}$$

$$y = \frac{\sin.\varphi}{n(i+1)}\left\{\begin{array}{c}\sin.(2i+1)\varphi + \sin.(2i-1)\varphi\\ + \sin.(2i-3)\varphi + \sin.(2i-5)\varphi\\ + \text{etc.}\ \ldots + \sin.\varphi\end{array}\right\}$$

Reductionibus igitur adhibitis colligetur fore

$$2n(i+1)x = \sin.(2i+2)\varphi$$

et

$$2n(i+1)y = 1 - \cos.(2i+2)\varphi,$$

unde patet curvam satisfacientem perpetuo manere circulum.

SCHOLION

34. Evidens autem est reliquis casibus omnibus solutiones methodo nostra datas maxime a se invicem esse discrepaturas atque adeo continuo ad altiores curvarum ordines esse ascensuras. Interim tamen, etiamsi solutio nostra infinitas praebeat curvas satisfacientes, nullum plane est dubium, quin praeter eas innumerabiles aliae revera assignari queant, quemadmodum pro casibus, quibus curvae debent esse rectificabiles, iam satis est ostensum. Eandem solutionum multiplicitatem insuper alio casu, quo $k = 3$, declarasse iuvabit.

EVOLUTIO CASUS QUO $k = 3$

35. Hoc quidem casu nostra methodus infinitas exhibet curvas algebraicas; verum praeter illas sequenti modo innumerabiles alias invenire licebit. Cum enim sit $\partial s = \frac{1}{n}\partial\varphi \sin.\varphi^2$, erit

$$\partial s = \frac{\partial\varphi}{2n}(1 - \cos.2\varphi),$$

quae formula nobis sequentes valores pro ∂x et ∂y assumendos suggerit

$$\partial x = \frac{\partial \varphi}{2n}(1 - \cos. 2\varphi) \cos. \lambda\varphi$$

et

$$\partial y = \frac{\partial \varphi}{2n}(1 - \cos. 2\varphi) \sin. \lambda\varphi,$$

quae formulae manifesto semper integrationem admittent solo casu $\lambda = \pm 2$ excepto. Quodsi enim reductiones notae in subsidium vocentur, proveniet

$$\frac{4n\partial x}{\partial \varphi} = 2 \cos. \lambda\varphi - \cos. (\lambda + 2)\varphi - \cos. (\lambda - 2)\varphi$$

et

$$\frac{4n\partial y}{\partial \varphi} = 2 \sin. \lambda\varphi - \sin. (\lambda + 2)\varphi - \sin. (\lambda - 2)\varphi,$$

quae ergo formulae integratae nobis praebent

$$4nx = + \frac{2 \sin. \lambda\varphi}{\lambda} - \frac{\sin. (\lambda + 2)\varphi}{\lambda + 2} - \frac{\sin. (\lambda - 2)\varphi}{\lambda - 2}$$

et

$$4ny = - \frac{2 \cos. \lambda\varphi}{\lambda} + \frac{\cos. (\lambda + 2)\varphi}{\lambda + 2} + \frac{\cos. (\lambda - 2)\varphi}{\lambda - 2}.$$

36. Quoniam hic pro λ non solum omnes numeros integros, verum etiam omnes fractiones assumere licet, evidens est istam solutionem infinities latius patere quam supra exhibitam. Quin etiam manifestum est istas novas solutiones omnes a superioribus penitus esse diversas.

37. Eodem modo casus tractari poterunt, quibus litterae k valor integer positivus quicunque tribuitur, propterea quod potestatem $\sin. \varphi^k$ semper in sinus vel cosinus simplices resolvere licet, quae partes deinde tam in $\sin. \lambda\varphi$ quam in $\cos. \lambda\varphi$ ductae evadent integrabiles, dummodo $\lambda\varphi$ non tale sit multiplum ipsius φ, cuiusmodi ex illa resolutione sunt natae.

38. Quoniam haec maxime sunt generalia atque ob hanc ipsam causam maiori illustratione indigeant, referamus formulas supra inventas ad casum quempiam specialem et in curvas algebraicas inquiramus, quarum arcus sive per arcum curvae elasticae $\int \frac{\partial v}{\sqrt{(1 - v^4)}}$ sive per applicatam eiusdem curvae $\int \frac{vv\partial v}{\sqrt{(1 - v^4)}}$ exprimatur.

EXEMPLUM 1

39. *Invenire curvas algebraicas, quarum arcus sit*

$$s = \int \frac{\partial v}{\sqrt{(1 - v^4)}}.$$

Cum igitur hic sit $m = 1$ et $n = 2$, erit $k = \frac{1}{2}$, unde solutionum specialium supra datarum prima nobis praebebit

$$x = \cos. \frac{1}{2}\varphi \sqrt{\sin. \varphi} \quad \text{et} \quad y = \sin. \frac{1}{2}\varphi \sqrt{\sin. \varphi}.$$

Quo nunc hinc angulum φ eliminemus, quaeramus

$$xx + yy = \sin. \varphi \quad \text{et} \quad 2xy = 2 \sin. \varphi \sin. \frac{1}{2}\varphi \cos. \frac{1}{2}\varphi = \sin. \varphi^2$$

eritque

$$2xy = (xx + yy)^2,$$

quae ergo curva est ordinis quarti et sub nomine Lemniscatae cognita, cuius adeo omnes arcus pari modo, quo circulares, inter se comparari posse iam dudum a Geometris est ostensum.

40. Simili modo sequentes solutiones speciales perducent ad alias curvas algebraicas eiusdem indolis, quae autem ad multo altiores ordines assurgent, quas hic idcirco fusius evolvere superfluum foret.

EXEMPLUM 2

41. *Invenire formulam algebraicam, cuius arcus sit*

$$s = \int \frac{vv\partial v}{\sqrt{(1 - v^4)}}.$$

Hic igitur est $m = 3$ et $n = 2$ ideoque $k = \frac{3}{2}$, unde species prima praebet

$$x = \frac{1}{3} \sin. \varphi^{\frac{3}{2}} \cos. \frac{3}{2}\varphi \quad \text{et} \quad y = \frac{1}{3} \sin. \varphi^{\frac{3}{2}} \sin. \frac{3}{2}\varphi,$$

unde erit

$$9(xx + yy) = \sin. \varphi^3 \quad \text{et} \quad 18xy = 2 \sin. \varphi^3 \sin. \frac{3}{2}\varphi \cos. \frac{3}{2}\varphi = \sin. \varphi^3 \sin. 3\varphi.$$

Cum igitur sit $\sin. 3\varphi = 3\sin.\varphi - 4\sin.\varphi^3$, erit

$$18xy = 3\sin.\varphi^4 - 4\sin.\varphi^6,$$

hinc porro

$$3\sin.\varphi^4 = 18xy + 324(xx+yy)^2 \quad \text{sive} \quad \sin.\varphi^4 = 6xy + 108(xx+yy)^2.$$

Hinc igitur deducimus binos valores pro $\sin.\varphi^{12}$, unde nascitur sequens aequatio

$$216(xy + 18(xx+yy)^2)^3 = 9^4(xx+yy)^4,$$

quae aequatio assurgit ad ordinem duodecimum videturque esse simplicissima, quae huic conditioni satisfaciat.

SCHOLION

42. Principia autem, quae hic stabilivimus, quaestionibus multo magis complicatis resolvendis sufficiunt, quemadmodum in sequenti problemate adhuc sumus ostensuri.

PROBLEMA MAGIS GENERALE

43. *Invenire curvas algebraicas, quarum arcus indefiniti s ita exprimantur, ut sit*

$$s = \int\frac{v^{m-1}\partial v}{\sqrt{(1-v^{2n})}}(a + bv^{2n} + cv^{4n} + dv^{6n} + \text{etc.})$$

SOLUTIO

Quotcunque terminos ista expressio contineat, sufficiet solutionem ad tres terminos accommodasse, quandoquidem hinc facile perspicietur, quomodo calculum ad quotcunque terminos extendi oporteat. Statuamus igitur ut ante

$$v^n = \sin.\varphi$$

ac posito $\frac{m}{n} = k$, quia inde fit

$$\frac{v^{m-1}\partial v}{\sqrt{(1-v^{2n})}} = \frac{1}{n}\partial\varphi\sin.\varphi^{k-1},$$

pro nostro problemate habebimus

$$\partial s = \frac{1}{n}\partial\varphi \sin.\varphi^{k-1}(a + b \sin.\varphi^2 + c \sin.\varphi^4).$$

44. Cum nunc hic habeamus tres partes, in quibus exponentes ipsius $\sin.\varphi$ sunt $k-1$, $k+1$, $k+3$, qui binario ascendunt, ponamus pro parte secunda $k+2=k'$ ac pro tertia $k+4=k''$, ut ternae nostrae partes fiant

$$\partial s = \frac{a\partial\varphi}{n}\sin.\varphi^{k-1} + \frac{b\partial\varphi}{n}\sin.\varphi^{k'-1} + \frac{c\partial\varphi}{n}\sin.\varphi^{k''-1}.$$

Has igitur singulatim multiplicemus per cosinum et sinum eiusdem anguli $(k+2i+1)\varphi$, qui pro parte secunda erit $(k'+2i-1)\varphi$, pro tertia autem $(k''+2i-3)\varphi$; ubi tantum notari oportet numerum integrum i ita accipi debere, ut ultimus numerus $2i-3$ maneat positivus.

45. His igitur constitutis ex formula ∂s prorsus ut supra determinare licebit elementa coordinatarum ∂x et ∂y, ponendo scilicet

$$\partial x = \partial s \cos.(k+2i+1)\varphi \quad \text{et} \quad \partial y = \partial s \sin.(k+2i+1)\varphi,$$

quandoquidem hinc fiet $\partial x^2 + \partial y^2 = \partial s^2$, unde ternis partibus pro ∂s scribendis ipsae coordinatae ita exprimentur

$$x = \left\{\begin{array}{l} \frac{a}{n}\int\partial\varphi \sin.\varphi^{k-1} \cos.(k+2i+1)\varphi \\ +\frac{b}{n}\int\partial\varphi \sin.\varphi^{k'-1} \cos.(k'+2i-1)\varphi \\ +\frac{c}{n}\int\partial\varphi \sin.\varphi^{k''-1} \cos.(k''+2i-3)\varphi \end{array}\right\}$$

$$y = \left\{\begin{array}{l} \frac{a}{n}\int\partial\varphi \sin.\varphi^{k-1} \sin.(k+2i+1)\varphi \\ +\frac{b}{n}\int\partial\varphi \sin.\varphi^{k'-1} \sin.(k'+2i-1)\varphi \\ +\frac{c}{n}\int\partial\varphi \sin.\varphi^{k''-1} \sin.(k''+2i-3)\varphi \end{array}\right\}$$

Ubi integralia singularum partium per formulas supra § 12 et § 17 exhi-

bitas assignare licet, siquidem ibi dedimus integralia harum formularum

$$\int \partial s \sin.(k+2i+1)\varphi \quad \text{et} \quad \int \partial s \cos.(k+2i+1)\varphi$$

existente $\partial s = \frac{\partial \varphi}{n} \sin.\varphi^{k-1}$.

46. Cum igitur hic loco i innumerabiles numeros integros assumere liceat, manifestum est etiam pro hoc problemate infinitas exhiberi posse solutiones, si modo excipiantur casus illi singulares, quibus quispiam denominator evanescit, id quod evenit, quando k vel cyphrae vel numero negativo integro aequatur. Caeterum hoc problema exemplo particulari illustrasse iuvabit.

EXEMPLUM

47. *Invenire curvas algebraicas, pro quibus sit*

$$s = \int \frac{\partial v}{\sqrt{(1-vv)}}(a + bv^2 + cv^4).$$

Hic ergo erit

$$m = 1, \quad n = 1 \quad \text{et} \quad k = 1 \quad \text{ideoque} \quad k' = 3 \quad \text{et} \quad k'' = 5,$$

quamobrem ambae coordinatae in genere ita exprimentur

$$x = \left\{ \begin{array}{l} a\int \partial\varphi \cos.(2i+2)\varphi \\ + b\int \partial\varphi \sin.\varphi^2 \cos.(2i+2)\varphi \\ + c\int \partial\varphi \sin.\varphi^4 \cos.(2i+2)\varphi \end{array} \right\}$$

$$y = \left\{ \begin{array}{l} a\int \partial\varphi \sin.(2i+2)\varphi \\ + b\int \partial\varphi \sin.\varphi^2 \sin.(2i+2)\varphi \\ + c\int \partial\varphi \sin.\varphi^4 \sin.(2i+2)\varphi \end{array} \right\}$$

Ubi autem notandum est numerum i unitate maiorem capi debere, ne $2i-3$ fiat negativum.

48. Quo igitur curvam simplicissimam satisfacientem nanciscamur, sumamus $i=2$ atque formulae integrales pro coordinatis erunt

$$x = \left\{ \begin{aligned} & a\int \partial\varphi \cos. 6\varphi \\ & + b\int \partial\varphi \sin. \varphi^2 \cos. 6\varphi \\ & + c\int \partial\varphi \sin. \varphi^4 \cos. 6\varphi \end{aligned} \right\}$$

et

$$y = \left\{ \begin{aligned} & a\int \partial\varphi \sin. 6\varphi \\ & + b\int \partial\varphi \sin. \varphi^2 \sin. 6\varphi \\ & + c\int \partial\varphi \sin. \varphi^4 \sin. 6\varphi \end{aligned} \right\}$$

Iam pro primis partibus est $k = 1$ et $\partial s = \partial\varphi$, unde erit

$$\int \partial s \cos. 6\varphi = \int \partial s \cos. (k+5)\varphi = \frac{\sin. \varphi}{3}(\cos. 5\varphi + \cos. 3\varphi + \cos. \varphi)$$

et

$$\int \partial s \sin. 6\varphi = \int \partial s \sin. (k+5)\varphi = \frac{\sin. \varphi}{3}(\sin. 5\varphi + \sin. 3\varphi + \sin. \varphi),$$

qui valores reducti dabunt

$$\int \partial s \cos. 6\varphi = \frac{1}{6} \sin. 6\varphi \quad \text{et} \quad \int \partial s \sin. 6\varphi = \frac{1}{6}(1 - \cos. 6\varphi),$$

quas formulas per quantitatem a multiplicari oportet.

49. Pro partibus secundis habemus $\partial s = \partial\varphi \sin. \varphi^2$ et $k = 3$, unde nanciscimur

$$\int \partial\varphi \sin. \varphi^2 \cos. 6\varphi = \int \partial s \cos. (k+3)\varphi = \frac{\sin. \varphi^3}{4}\left(\cos. 5\varphi + \frac{1}{3}\cos. 3\varphi\right)$$

et

$$\int \partial\varphi \sin. \varphi^2 \sin. 6\varphi = \int \partial s \sin. (k+3)\varphi = \frac{\sin. \varphi^3}{4}\left(\sin. 5\varphi + \frac{1}{3}\sin. 3\varphi\right).$$

Prior forma ob $\sin. \varphi^3 = \frac{3}{4}\sin. \varphi - \frac{1}{4}\sin. 3\varphi$ transit in hanc

$$\int \partial s \cos. 6\varphi$$

$$= \frac{1}{16}\left(3 \sin. \varphi \cos. 5\varphi - \sin. 3\varphi \cos. 5\varphi + \sin. \varphi \cos. 3\varphi - \frac{1}{3}\sin. 3\varphi \cos. 3\varphi\right)$$

ideoque

$$\int \partial s \cos. 6\varphi = \frac{1}{32}\left(-2\sin. 4\varphi + \frac{8}{3}\sin. 6\varphi - 1 \sin. 8\varphi\right)$$

$$= -\frac{1}{16}\sin. 4\varphi + \frac{1}{12}\sin. 6\varphi - \frac{1}{32}\sin. 8\varphi.$$

Simili modo habebimus

$$\int \partial s \sin. 6\varphi$$
$$= \frac{1}{16}\left(3 \sin. \varphi \sin. 5\varphi - \sin. 3\varphi \sin. 5\varphi + \sin. \varphi \sin. 3\varphi - \frac{1}{3} \sin. 3\varphi^2\right)$$

ideoque

$$\int \partial s \sin. 6\varphi = \frac{1}{32}\left(2 \cos. 4\varphi - \frac{8}{3} \cos. 6\varphi + \cos. 8\varphi\right)$$
$$= +\frac{1}{16} \cos. 4\varphi - \frac{1}{12} \cos. 6\varphi + \frac{1}{32} \cos. 8\varphi.^{1)}$$

50. Verum in hoc negotio formulis supra datis penitus carere possumus; cum enim sit

$$\sin. \varphi^2 = \frac{1}{2} - \frac{1}{2} \cos. 2\varphi,$$

erit primo pro partibus secundis littera b affectis

$$\int \partial\varphi \sin. \varphi^2 \cos. 6\varphi = \frac{1}{2}\int \partial\varphi \cos. 6\varphi(1 - \cos. 2\varphi)$$
$$= \frac{1}{2}\int \partial\varphi\left(\cos. 6\varphi - \frac{1}{2} \cos. 8\varphi - \frac{1}{2} \cos. 4\varphi\right),$$

cuius integrale manifesto est

$$= \frac{1}{12} \sin. 6\varphi - \frac{1}{32} \sin. 8\varphi - \frac{1}{16} \sin. 4\varphi.$$

Deinde ob

$$\sin. \varphi^2 \sin. 6\varphi = \frac{1}{2} \sin. 6\varphi - \frac{1}{2} \cos. 2\varphi \sin. 6\varphi$$
$$= \frac{1}{2} \sin. 6\varphi - \frac{1}{4} \sin. 8\varphi - \frac{1}{4} \sin. 4\varphi$$

habebimus

$$\int \partial s \sin. 6\varphi = -\frac{1}{12} \cos. 6\varphi + \frac{1}{32} \cos. 8\varphi + \frac{1}{16} \cos. 4\varphi,$$

quas formulas per litteram b multiplicari oportet.

51. Denique pro tertiis partibus littera c affectis cum sit

$$\sin. \varphi^4 = \frac{3}{8} - \frac{1}{2} \cos. 2\varphi + \frac{1}{8} \cos. 4\varphi,$$

1) EULERUS hic omisit constantem $-\frac{1}{96}$. A. K.

erit

$$\sin.\varphi^4 \cos.6\varphi = \frac{3}{8}\cos.6\varphi - \frac{1}{4}\cos.8\varphi - \frac{1}{4}\cos.4\varphi + \frac{1}{16}\cos.10\varphi + \frac{1}{16}\cos.2\varphi,$$

unde integrando nanciscimur

$$\int \partial\varphi \sin.\varphi^4 \cos.6\varphi$$
$$= \frac{1}{16}\sin.6\varphi - \frac{1}{32}\sin.8\varphi - \frac{1}{16}\sin.4\varphi + \frac{1}{160}\sin.10\varphi + \frac{1}{32}\sin.2\varphi.$$

Deinde vero erit

$$\sin.\varphi^4 \sin.6\varphi = \frac{3}{8}\sin.6\varphi - \frac{1}{4}\sin.8\varphi - \frac{1}{4}\sin.4\varphi + \frac{1}{16}\sin.10\varphi + \frac{1}{16}\sin.2\varphi$$

ideoque integrando habebimus

$$\int \partial\varphi \sin.\varphi^4 \sin.6\varphi$$
$$= -\frac{1}{16}\cos.6\varphi + \frac{1}{32}\cos.8\varphi + \frac{1}{16}\cos.4\varphi - \frac{1}{160}\cos.10\varphi - \frac{1}{32}\cos.2\varphi.$$

52. His igitur colligendis ambae coordinatae x et y sequenti modo expressae reperiuntur

$$x = \frac{c}{32}\sin.2\varphi - \frac{b+c}{16}\sin.4\varphi + \left(\frac{a}{6} + \frac{b}{12} + \frac{c}{16}\right)\sin.6\varphi$$
$$- \frac{b+c}{32}\sin.8\varphi + \frac{c}{160}\sin.10\varphi,$$

$$y = -\frac{c}{32}\cos.2\varphi + \frac{b+c}{16}\cos.4\varphi - \left(\frac{a}{6} + \frac{b}{12} + \frac{c}{16}\right)\cos.6\varphi$$
$$+ \frac{b+c}{32}\cos.8\varphi - \frac{c}{160}\cos.10\varphi,$$

ubi constantem $\frac{a}{6}$ in prima parte pro y ingressam omisimus.

METHODUS SUCCINCTIOR COMPARATIONES QUANTITATUM TRANSCENDENTIUM IN FORMA
$$\int \frac{P\partial z}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}}$$
CONTENTARUM INVENIENDI

M. S. Academiae exhib. die 3 Novembris 1777
Commentatio 676 indicis ENESTROEMIANI
Institutiones calculi integralis 4, 1794, p. 504—524

In Capite VI Sect. II *Institutionum* mearum *Calculi Integralis* Tom. I[1]) insignes tradidi comparationes inter quantitates maxime transcendentes, ad quas deductus eram methodo penitus indirecta. Postquam igitur non ita pridem illustris DE LA GRANGE[2]) methodum maxime ingeniosam excogitasset easdem comparationes inveniendi, totum hoc argumentum multo succinctius et elegantius tractari poterit, quam mihi quidem tum temporis licebat, unde sequentia Supplementa Geometris haud displicebunt.

HYPOTHESIS 1

80. Denotet hic perpetuo character $\Pi : z$ valorem formulae integralis
$$\int \frac{\partial z}{\sqrt{(\alpha + \beta z + \gamma zz + \delta z^3 + \varepsilon z^4)}}$$
ita sumtae, ut evanescat posito $z = 0$. Ponatur autem brevitatis gratia
$$\alpha + \beta z + \gamma zz + \delta z^3 + \varepsilon z^4 = Z,$$

1) L. EULERI, *Institutionum calculi integralis volumen primum, in quo methodus integrandi a primis principiis usque ad integrationem aequationum differentialium primi gradus pertractatur.* Petropoli 1768; *LEONHARDI EULERI Opera omnia,* series I, vol. 11. A. K.

2) Vide notam p. 1. A. K.

ita ut sit

$$\Pi : z = \int \frac{\partial z}{\sqrt{Z}}.$$

Tum vero concipiatur super axe oz (Fig. 1) exstructa eiusmodi curva OZ,

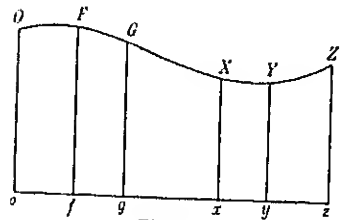

Fig. 1.

cuius singuli arcus OZ abscissis $oz = z$ respondentes exprimantur per formulam $\Pi : z = \int \frac{\partial z}{\sqrt{Z}}$, atque haec curva ista insigni proprietate erit praedita, ut sumto in ea pro lubitu arcu quocunque FG a quovis alio puncto X semper arcus XY illi arcui FG aequalis geometrice abscindi possit, cuius demonstrationem solutio sequentis problematis suppeditabit.

PROBLEMA 1

81. *Si in curva modo descripta proponatur arcus quicunque FG, innumerabiles alios arcus XY in eadem curva geometrice assignare, qui singuli eidem arcui FG sint aequales.*

SOLUTIO

Ductis ex punctis F et G ad axem oz applicatis Ff et Gg vocentur abscissae $of = f$ et $og = g$ eruntque arcus $OF = \Pi : f$ et $OG = \Pi : g$, unde longitudo arcus propositi FG erit $= \Pi : g - \Pi : f$. Simili modo pro quovis arcu quaesito XY vocentur abscissae $ox = x$ et $oy = y$ eruntque arcus $OX = \Pi : x$ et $OY = \Pi : y$ ideoque arcus $XY = \Pi : y - \Pi : x$; qui cum aequalis esse debeat arcui FG, habebitur ista aequatio

$$\Pi : y - \Pi : x = \Pi : g - \Pi : f,$$

cui satisfieri oportet.

82. Quoniam puncta F et G considerantur ut fixa, dum puncta X et Y per totam curvam variari possunt, differentiatio nobis praebebit hanc aequationem $\partial \Pi : y - \partial \Pi : x = 0$. Quare, cum sit per hypothesin

$$\Pi : x = \int \frac{\partial x}{\sqrt{X}} \quad \text{et} \quad \Pi : y = \int \frac{\partial y}{\sqrt{Y}}$$

existente

$$X = \alpha + \beta x + \gamma xx + \delta x^3 + \varepsilon x^4 \quad \text{et} \quad Y = \alpha + \beta y + \gamma yy + \delta y^3 + \varepsilon y^4,$$

solutio problematis perducta est ad hanc aequationem differentialem

$$\frac{\partial y}{\sqrt{Y}} - \frac{\partial x}{\sqrt{X}} = 0.$$

83. Hic iam methodum ill. DE LA GRANGE in subsidium vocantes statuamus

$$\frac{\partial x}{\sqrt{X}} = \partial t$$

eritque $\frac{\partial y}{\sqrt{Y}} = \partial t$. Hic scilicet novum elementum ∂t in calculum introducimus, quod in sequentibus differentiationibus ut constans tractetur; tum igitur habebimus

$$\frac{\partial x}{\partial t} = \sqrt{X} \quad \text{et} \quad \frac{\partial y}{\partial t} = \sqrt{Y}.$$

Quodsi ergo porro statuamus

$$y + x = p \quad \text{et} \quad y - x = q,$$

habebimus hinc

$$\frac{\partial p}{\partial t} = \sqrt{Y} + \sqrt{X} \quad \text{et} \quad \frac{\partial q}{\partial t} = \sqrt{Y} - \sqrt{X},$$

quarum formularum productum praebet

$$\frac{\partial p \partial q}{\partial t^2} = Y - X.$$

Valoribus ergo loco Y et X substitutis erit

$$\frac{\partial p \partial q}{\partial t^2} = \beta(y - x) + \gamma(y^2 - x^2) + \delta(y^3 - x^3) + \varepsilon(y^4 - x^4).$$

Quare, cum sit

$$y = \frac{p+q}{2} \quad \text{et} \quad x = \frac{p-q}{2},$$

erit

$$y - x = q, \quad y^2 - x^2 = pq, \quad y^3 - x^3 = \frac{1}{4}q(3pp + qq)$$

et

$$y^4 - x^4 = \frac{1}{2}pq(pp + qq),$$

quibus substitutis factaque divisione per q habebitur

$$\frac{\partial p \partial q}{q \partial t^2} = \beta + \gamma p + \frac{1}{4}\delta(3pp + qq) + \frac{1}{2}\varepsilon p(pp + qq),$$

cuius aequationis plurimus erit usus in sequenti calculo.

84. Iam sumtis quadratis primae aequationes dabunt

$$\frac{\partial x^2}{\partial t^2} = X \quad \text{et} \quad \frac{\partial y^2}{\partial t^2} = Y,$$

quae denuo differentientur, quem in finem ponamus brevitatis gratia

$$\partial X = X' \partial x \quad \text{et} \quad \partial Y = Y' \partial y$$

atque hinc nanciscemur

$$\frac{2\partial\partial x}{\partial t^2} = X' \quad \text{et} \quad \frac{2\partial\partial y}{\partial t^2} = Y',$$

quibus additis erit

$$\frac{2\partial\partial p}{\partial t^2} = X' + Y'.$$

Cum igitur sit

$$X' = \beta + 2\gamma x + 3\delta xx + 4\varepsilon x^3 \quad \text{et} \quad Y' = \beta + 2\gamma y + 3\delta yy + 4\varepsilon y^3,$$

erit

$$\frac{2\partial\partial p}{\partial t^2} = 2\beta + 2\gamma(x+y) + 3\delta(x^2+y^2) + 4\varepsilon(x^3+y^3).$$

Introducendo igitur litteras p et q ut ante fiet

$$x + y = p, \quad x^2 + y^2 = \frac{1}{2}(pp + qq), \quad x^3 + y^3 = \frac{1}{4}p(pp + 3qq)$$

sicque ista aequatio hanc induet formam

$$\frac{2\partial\partial p}{\partial t^2} = 2\beta + 2\gamma p + \frac{3}{2}\delta(pp + qq) + \varepsilon p(pp + 3qq).$$

85. Ab hac iam postrema aequatione subtrahatur praecedens bis sumta ac remanebit

$$\frac{2\partial\partial p}{\partial t^2} - \frac{2\partial p \partial q}{q \partial t^2} = \delta qq + 2\varepsilon pqq.$$

Hinc per qq dividendo habebimus

$$\frac{1}{\partial t^2}\left(\frac{2\partial\partial p}{qq}-\frac{2\partial p\,\partial q}{q^3}\right)=\delta+2\varepsilon p,$$

cuius utrumque membrum manifesto integrationem admittit, si ducatur in elementum ∂p. Hoc enim facto aequatio integralis erit

$$\frac{\partial p^2}{qq\partial t^2}=C+\delta p+\varepsilon pp.$$

86. Initio autem vidimus esse $\frac{\partial p}{\partial t}=\sqrt{X}+\sqrt{Y}$ hincque statim pervenimus ad aequationem integralem algebraicam hanc

$$\frac{(\sqrt{X}+\sqrt{Y})^2}{qq}=C+\delta p+\varepsilon pp.$$

Quare cum sit $p=x+y$ et $q=y-x$, haec aequatio evoluta fiet

$$\frac{X+Y+2\sqrt{XY}}{(y-x)^2}=C+\delta(x+y)+\varepsilon(x+y)^2,$$

ubi constantem per integrationem ingressam secundum indolem problematis ita definiri oportet, ut, dum punctum X incidit in punctum F, punctum Y in ipsum punctum G cadat, sive ut facto $x=f$ fiat $y=g$.

87. Cum iam sit

$$X+Y=2\alpha+\beta(x+y)+\gamma(x^2+y^2)+\delta(x^3+y^3)+\varepsilon(x^4+y^4),$$

si terminos $\delta(x+y)+\varepsilon(x+y)^2$ in alteram partem transferimus, perveniemus ad hanc aequationem

$$\frac{2\alpha+\beta(x+y)+\gamma(x^2+y^2)+\delta xy(x+y)+2\varepsilon xxyy+2\sqrt{XY}}{(y-x)^2}=C.$$

Subtrahamus autem insuper utrinque γ et loco $C-\gamma$ scribamus Δ hocque modo nostra aequatio reducetur ad hanc formam satis concinnam

$$\frac{2\alpha+\beta(x+y)+2\gamma xy+\delta xy(x+y)+2\varepsilon xxyy+2\sqrt{XY}}{(y-x)^2}=\Delta.$$

88. Quia nunc $\varDelta$ ita determinari debet, ut sumto $x=f$ fiat $y=g$, si secundum analogiam statuamus

$$\alpha+\beta f+\gamma ff+\delta f^3+\varepsilon f^4=F \quad \text{et} \quad \alpha+\beta g+\gamma gg+\delta g^3+\varepsilon g^4=G,$$

erit ista constans $\varDelta$ ita expressa

$$\varDelta=\frac{2\alpha+\beta(f+g)+2\gamma fg+\delta fg(f+g)+2\varepsilon ffgg+2\sqrt{FG}}{(g-f)^2}.$$

Hac igitur aequatione inventa si ipsi x pro lubitu tribuatur valor quicunque, inde elici poterit valor ipsius y, ita ut alter terminus X arcus quaesiti XY pro arbitrio assumi possit. Verum facile patet istam determinationem in calculos perquam molestos praecipitare, quandoquidem aequatio inventa quadratis sumendis ab irrationalitate $\sqrt{XY}$ liberari deberet. Sequenti autem modo ista investigatio sublevari poterit.

89. Quoniam ista formula

$$2\alpha+\beta(x+y)+2\gamma xy+\delta xy(x+y)+2\varepsilon xxyy$$

essentialiter in calculum ingreditur, eius loco brevitatis gratia scribamus hunc characterem $[x, y]$, cuius ergo valor erit cognitus, etiam si loco x et y aliae litterae accipiantur. Hoc igitur modo aequatio inventa ita referri poterit

$$\frac{[x, y]+2\sqrt{XY}}{(y-x)^2}=\frac{[f, g]+2\sqrt{FG}}{(g-f)^2},$$

quae ergo aequatio exprimit relationem inter binas ordinatas x et y, ut problemati satisfiat, hoc est, ut fiat

$$\Pi:y-\Pi:x=\Pi:g-\Pi:f.$$

Quare cum hinc etiam sequatur

$$\Pi:y-\Pi:g=\Pi:x-\Pi:f,$$

aequatio hinc ista exsurget

$$\frac{[g, y]+2\sqrt{GY}}{(y-g)^2}=\frac{[f, x]+2\sqrt{FX}}{(x-f)^2}.$$

90. Ex hac iam aequatione cum priore coniuncta facile eliminari poterit formula radicalis $\sqrt{Y}$ sicque aequatio habebitur tantum litteram y tanquam incognitam involvens, unde eius valor haud difficulter definiri potest. Calculum autem hunc instituenti patebit tantum ad aequationem quadraticam perveniri, ita ut bini valores pro puncto Y reperiantur, quemadmodum rei natura postulat, dum sumto puncto X alterum punctum Y tam dextrorsum quam sinistrorsum cadere poterit. Hinc autem calculo fusius non immoramur, quandoquidem hic potissimum est propositum totam huius problematis solutionem per methodum directam a priori repetere.

HYPOTHESIS 2

91. Constituta super axe oz (Fig. 2) curva OZ in priori hypothesi descripta concipiatur super eodem axe alia curva insuper descripta $\mathfrak{O}\mathfrak{Z}$ ita comparata, ut abscissae $oz = z$ respondeat arcus $\mathfrak{O}\mathfrak{Z} = \Phi : z$, ita ut sit

$$\Phi : z = \int \frac{\partial z(\mathfrak{A} + \mathfrak{B}z + \mathfrak{C}zz + \mathfrak{D}z^3 + \text{etc.})}{\sqrt{Z}}$$

integrali hoc pariter ita sumto, ut evanescat posito $z = 0$, existente ut ante

$$Z = \alpha + \beta z + \gamma zz + \delta z^3 + \varepsilon z^4.$$

Pro numeratore autem ponamus brevitatis gratia

$$\mathfrak{A} + \mathfrak{B}z + \mathfrak{C}zz + \mathfrak{D}z^3 + \text{etc.} = \mathfrak{Z},$$

ita ut sit

$$\Phi : z = \int \frac{\mathfrak{Z}\partial z}{\sqrt{Z}}.$$

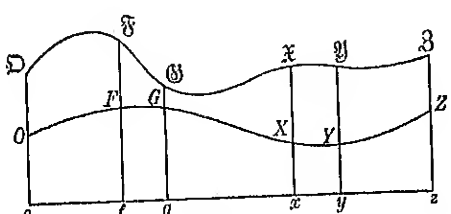

Fig. 2.

92. Ista iam curva hac ratione descripta hac insigni proprietate erit praedita, ut, si in priore curva rescissi fuerint arcus FG et XY inter se aequales, productis iisdem applicatis in nova curva arcuum hoc modo rescissorum $\mathfrak{F}\mathfrak{G}$ et $\mathfrak{X}\mathfrak{Y}$ differentia vel algebraice vel saltem per logarithmos et arcus circulares assignari possit, cuius rei veritatem solutio sequentis problematis demonstrabit.

PROBLEMA 2

93. *Si in curva secundum primam hypothesin descripta abscissi fuerint duo arcus aequales FG et XY iisque in curva modo descripta respondeant arcus $\mathfrak{FG}$ et $\mathfrak{XY}$, quibus scilicet eaedem abscissae in axe conveniant, differentiam inter hos binos arcus investigare.*

SOLUTIO

Quia igitur hic quaeritur differentia inter arcus $\mathfrak{FG}$ et $\mathfrak{XY}$, ponatur ea $= V$, quae ergo spectari poterit tanquam certa functio ipsarum x et y, si quidem puncta $\mathfrak{F}$ et $\mathfrak{G}$ tanquam fixa consideramus. Cum igitur sit

$$\text{arcus } \mathfrak{FG} = \Phi : g - \Phi : f \quad \text{et} \quad \text{arcus } \mathfrak{XY} = \Phi : y - \Phi : x,$$

habebimus

$$\Phi : y - \Phi : x = \Phi : g - \Phi : f + V,$$

unde differentiando habebimus

$$\frac{\mathfrak{Y}\partial y}{\sqrt{Y}} - \frac{\mathfrak{X}\partial x}{\sqrt{X}} = \partial V,$$

quia litteras f et g pro constantibus habemus.

94. Ponamus nunc, ut supra factum est,

$$\frac{\partial x}{\sqrt{X}} = \frac{\partial y}{\sqrt{Y}} = \partial t$$

et haec aequatio induet istam formam

$$(\mathfrak{Y} - \mathfrak{X})\partial t = \partial V.$$

Verum in solutione primi problematis deducti fuimus ad hanc aequationem finalem

$$\frac{\partial p^2}{qq\partial t^2} = C + \delta p + \varepsilon pp,$$

unde fit

$$\frac{\partial p}{q\partial t} = \sqrt{(C + \delta p + \varepsilon pp)} = \sqrt{(\varDelta + \gamma + \delta p + \varepsilon pp)},$$

atque hinc colligimus

$$\partial t = \frac{\partial p}{q\sqrt{(\varDelta + \gamma + \delta p + \varepsilon pp)}},$$

ubi est $p = x + y$ et $q = y - x$. Hoc ergo valore inducto aequatio differentialis resolvenda est

$$\partial V = \frac{(\mathfrak{Y} - \mathfrak{X})\partial p}{q\sqrt{(\varDelta + \gamma + \delta p + \varepsilon pp)}},$$

ubi est

$$\mathfrak{X} = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}xx + \mathfrak{D}x^3 + \text{etc.}$$

similique modo

$$\mathfrak{Y} = \mathfrak{A} + \mathfrak{B}y + \mathfrak{C}yy + \mathfrak{D}y^3 + \text{etc.},$$

quousque libuerit continuando.

95. Quodsi iam hos valores substituamus, habebimus

$$\mathfrak{Y} - \mathfrak{X} = \mathfrak{B}(y - x) + \mathfrak{C}(y^2 - x^2) + \mathfrak{D}(y^3 - x^3) + \mathfrak{E}(y^4 - x^4) + \text{etc.},$$

unde, si loco x et y introducamus quantitates p et q, ob $x = \frac{p-q}{2}$ et $y = \frac{p+q}{2}$ orientur sequentes valores

$$y - x = q, \quad y^2 - x^2 = pq, \quad y^3 - x^3 = \tfrac{1}{4}q(3pp + qq),$$

$$y^4 - x^4 = \tfrac{1}{2}pq(pp + qq), \quad y^5 - x^5 = \tfrac{1}{16}q(5p^4 + 10ppqq + q^4) \quad \text{etc.}$$

96. Quantitas ergo V per sequentes formulas integrales secundum numerum litterarum $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. determinatur

$$\begin{aligned} V = {} & \mathfrak{B}\int \frac{\partial p}{\sqrt{(\varDelta + \gamma + \delta p + \varepsilon pp)}} + \mathfrak{C}\int \frac{p\,\partial p}{\sqrt{(\varDelta + \gamma + \delta p + \varepsilon pp)}} \\ & + \tfrac{1}{4}\mathfrak{D}\int \frac{(3pp + qq)\partial p}{\sqrt{(\varDelta + \gamma + \delta p + \varepsilon pp)}} + \tfrac{1}{2}\mathfrak{E}\int \frac{p(pp + qq)\partial p}{\sqrt{(\varDelta + \gamma + \delta p + \varepsilon pp)}} \\ & + \tfrac{1}{16}\mathfrak{F}\int \frac{(5p^4 + 10ppqq + q^4)\partial p}{\sqrt{(\varDelta + \gamma + \delta p + \varepsilon pp)}} + \text{etc.} \end{aligned}$$

Quarum formularum duae priores iam absolute exhiberi possunt, sive algebraice, quod evenit, si $\varepsilon = 0$, sive per logarithmos, si valor ipsius ε fuerit positivus, sive per arcus circulares, si valor ipsius ε fuerit negativus. Reliquae vero formulae exigunt relationem inter p et q, quam deinceps investigabimus. Hic tantum notetur potestates solas pares ipsius q in has formulas ingredi.

97. Hic autem littera A eundem valorem constantem designat, quem supra iam definivimus, qui erat

$$A = \frac{2\alpha + \beta(f+g) + 2\gamma fg + \delta fg(f+g) + 2\varepsilon ffgg + 2\sqrt{FG}}{(g-f)^2}.$$

Praeterea vero cum esse debeat

$$\Pi:y - \Pi:x = \Pi:g - \Pi:f + V,$$

evidens est casu, quo $x = f$ et $y = g$, fieri debere $V = 0$; quamobrem formulae illae integrales pro V inventae ita capi debebunt, ut posito $p = f + g$ et $q = g - f$ valor ipsius V evanescat.

ANALYSIS PRO INVESTIGANDA RELATIONE INTER p ET q

98. Quia iam invenimus aequationem finitam inter x et y, ex ea quoque ponendo $y = \frac{p+q}{2}$ et $x = \frac{p-q}{2}$ relatio inter litteras p et q derivari posset; verum hoc calculos nimis taediosos postularet, quamobrem aliam viam ineamus istam relationem ex formulis differentialibus deducendi. Cum enim sit

$$\frac{\partial p}{\partial q} = \frac{\partial y + \partial x}{\partial y - \partial x},$$

ob proportionem

$$\partial x : \partial y = \sqrt{X} : \sqrt{Y}$$

erit

$$\frac{\partial p}{\partial q} = \frac{\sqrt{Y} + \sqrt{X}}{\sqrt{Y} - \sqrt{X}};$$

supra autem invenimus esse

$$\frac{\sqrt{Y} + \sqrt{X}}{q} = \sqrt{(A + \gamma + \delta p + \varepsilon pp)},$$

ubi A eandem denotat constantem, quam modo ante definivimus.

99. Nunc igitur fractio pro $\frac{\partial p}{\partial q}$ inventa supra et infra multiplicetur per $\sqrt{Y} + \sqrt{X}$, et cum sit

$$(\sqrt{Y} + \sqrt{X})^2 = qq(A + \gamma + \delta p + \varepsilon pp),$$

habebimus hanc aequationem

$$\frac{\partial p}{\partial q}=\frac{qq(\varDelta+\gamma+\delta p+\varepsilon pp)}{Y-X},$$

cuius denominatorem iam supra § 83 evolvimus, ubi invenimus esse

$$Y-X=\beta q+\gamma pq+\tfrac{1}{4}\delta q(3pp+qq)+\tfrac{1}{2}\varepsilon pq(pp+qq);$$

quo valore substituto erit

$$\frac{\partial p}{\partial q}=\frac{q(\varDelta+\gamma+\delta p+\varepsilon pp)}{\beta+\gamma p+\frac{1}{4}\delta(3pp+qq)+\frac{1}{2}\varepsilon p(pp+qq)},$$

quae reducitur ad hanc formam

$$2q\partial q=\frac{(2\beta+2\gamma p+\frac{1}{2}\delta(3pp+qq)+\varepsilon p(pp+qq))\partial p}{\varDelta+\gamma+\delta p+\varepsilon pp}.$$

100. Transferamus terminos, qui continent qq, a dextra in sinistram partem, ut obtineamus hanc aequationem

$$2q\partial q-\frac{qq\partial p(\frac{1}{2}\delta+\varepsilon p)}{\varDelta+\gamma+\delta p+\varepsilon pp}=\frac{(2\beta+2\gamma p+\frac{3}{2}\delta pp+\varepsilon p^3)\partial p}{\varDelta+\gamma+\delta p+\varepsilon pp}.$$

Membrum huius aequationis sinistrum integrabile reddi potest, si per certam functionem ipsius p, quae sit $=\Pi$, multiplicetur, quando fuerit

$$\frac{\partial \Pi}{\Pi}=-\frac{\partial p(\frac{1}{2}\delta+\varepsilon p)}{\varDelta+\gamma+\delta p+\varepsilon pp},$$

quae aequatio integrata dat

$$l\Pi=-\tfrac{1}{2}l(\varDelta+\gamma+\delta p+\varepsilon pp).$$

Sicque erit multiplicator iste

$$\Pi=\frac{1}{\sqrt{(\varDelta+\gamma+\delta p+\varepsilon pp)}};$$

tum autem integrale quaesitum erit

$$\frac{qq}{\sqrt{(\varDelta+\gamma+\delta p+\varepsilon pp)}}=\int\frac{(2\beta+2\gamma p+\frac{3}{2}\delta pp+\varepsilon p^3)\partial p}{(\varDelta+\gamma+\delta p+\varepsilon pp)^{\frac{3}{2}}}.$$

101. Hoc postremum integrale manifesto continet formam

$$\frac{pp}{\sqrt{(\varDelta+\gamma+\delta p+\varepsilon pp)}},$$

quippe cuius differentiale est

$$\frac{(2\varDelta p+2\gamma p+\frac{3}{2}\delta pp+\varepsilon p^3)\partial p}{(\varDelta+\gamma+\delta p+\varepsilon pp)^{\frac{3}{2}}};$$

quare integrale ita potest repraesentari

$$\frac{qq}{\sqrt{(\varDelta+\gamma+\delta p+\varepsilon pp)}}=\frac{pp}{\sqrt{(\varDelta+\gamma+\delta p+\varepsilon pp)}}+\int\frac{(2\beta-2\varDelta p)\partial p}{(\varDelta+\gamma+\delta p+\varepsilon pp)^{\frac{3}{2}}},$$

quod postremum integrale statuatur

$$=\frac{m+np}{\sqrt{(\varDelta+\gamma+\delta p+\varepsilon pp)}};$$

huius enim differentiale est

$$\frac{((\varDelta+\gamma)n-\frac{1}{2}\delta m+(\frac{1}{2}\delta n-\varepsilon m)p)\partial p}{(\varDelta+\gamma+\delta p+\varepsilon pp)^{\frac{3}{2}}}$$

ideoque fieri debet

$$(\varDelta+\gamma)n-\frac{1}{2}\delta m=2\beta \quad \text{et} \quad \frac{1}{2}\delta n-\varepsilon m=-2\varDelta,$$

unde deducuntur valores

$$m=\frac{4\beta\delta+8\varDelta\varDelta+8\varDelta\gamma}{4\varDelta\varepsilon+4\gamma\varepsilon-\delta\delta} \quad \text{et} \quad n=\frac{8\beta\varepsilon+4\varDelta\delta}{4\varDelta\varepsilon+4\gamma\varepsilon-\delta\delta},$$

quarum fractionum loco in calculo retineamus litteras m et n; consequenter adiecta constante aequatio integralis ita se habebit

$$qq=pp+np+m+C\sqrt{(\varDelta+\gamma+\delta p+\varepsilon pp)}.$$

102. Ista autem constans ita definiri debet, ut posito $p=f+g$ fiat $q=g-f$, ex quo quantitas illa constans ita determinabitur

$$C=-\frac{4fg+n(f+g)+m}{\sqrt{(\varDelta+\gamma+\delta(f+g)+\varepsilon(f+g)^2)}}.$$

Hoc ergo valore invento facile assignari poterunt valores non solum ipsius qq, sed etiam eius potestatum parium q^4, q^6, q^8 etc., quibus indigemus. Atque hinc intelligitur pro inveniendo valore ipsius V alias formulas integrales non

occurrere, nisi quae involvant quantitatem radicalem $\sqrt{(A+\gamma+\delta p+\varepsilon pp)}$, quarum ergo integratio, nisi algebraice institui queat, semper per logarithmos et arcus circulares expediri poterit. Evidens autem est casu, quo $\varepsilon=0$, omnia integralia algebraice exprimi posse.

103. Quodsi ergo pro priori curva OZ fuerit

$$\Pi : z = \int \frac{\partial z}{\sqrt{(\alpha+\beta z+\gamma z^2+\delta z^3)}},$$

pro altera vero curva

$$\Phi : z = \int \frac{\partial z(\mathfrak{A}+\mathfrak{B}z+\mathfrak{C}zz+\mathfrak{D}z^3+\text{etc.})}{\sqrt{(\alpha+\beta z+\gamma zz+\delta z^3)}},$$

tum sumtis in priori curva arcubus aequalibus FG et XY iis in altera curva respondebunt arcus $\mathfrak{F}\mathfrak{G}$ et $\mathfrak{X}\mathfrak{Y}$, quorum differentia semper geometrice assignari poterit. Interdum etiam fieri potest, ut differentia V in nihilum abeat, id quod quidem semper evenit sumto $x=f$.

104. Praeterea vero etiam datur alius casus maxime memorabilis, quod differentia illa V algebraice exprimi poterit, qui scilicet semper locum habebit, quando tam in denominatore quam in numeratore tantum potestates pares ipsius z occurrunt, hoc est, si fuerit pro curva priore

$$\Pi : z = \int \frac{\partial z}{\sqrt{(\alpha+\gamma zz+\varepsilon z^4)}},$$

pro altera vero curva

$$\Phi : z = \int \frac{\partial z(\mathfrak{A}+\mathfrak{C}zz+\mathfrak{E}z^4+\mathfrak{G}z^6+\text{etc.})}{\sqrt{(\alpha+\gamma zz+\varepsilon z^4)}}.$$

His enim casibus si in priore curva arcus aequales FG et XY abscindantur, tum arcuum in altera curva respondentium $\mathfrak{F}\mathfrak{G}$ et $\mathfrak{X}\mathfrak{Y}$ differentia semper algebraice seu geometrice exhiberi poterit, ad quotcunque terminos etiam numerator $\mathfrak{A}+\mathfrak{C}zz+\mathfrak{E}z^4+$ etc. continuetur, atque hic est casus, quem olim tam in *Calculo integrali*[1]) quam alibi[2]) fusius pertractavi.

1) Vide notam 1 p. 207. A. K.

2) L. EULERI Commentatio 261 (indicis ENESTROEMIANI): *Specimen alterum methodi novae quantitates transcendentes inter se comparandi; de comparatione arcuum ellipsis*, Novi comment. acad. sc. Petrop. 7 (1758/9), 1761, p. 3; *LEONHARDI EULERI Opera omnia*, series I, vol. 20. A. K.

105. Ad hoc ostendendum, quia habemus tam $\delta = 0$ quam $\beta = 0$, primo erit

$$qq = pp + m + C\sqrt{(\varDelta + \gamma + \varepsilon pp)},$$

ita ut hic tantum potestates pares ipsius p occurrant; tum autem pro litteris germanicis $\mathfrak{C}$, $\mathfrak{E}$, $\mathfrak{G}$ etc. formulae integrandae sequenti modo se habebunt:

Pro littera $\mathfrak{C}$

$$\int \frac{p\partial p}{\sqrt{(\varDelta + \gamma + \varepsilon pp)}},$$

quae per se est absolute integrabilis.

Pro littera $\mathfrak{E}$

$$\int \frac{p(pp + qq)\partial p}{\sqrt{(\varDelta + \gamma + \varepsilon pp)}},$$

quae loco qq substituto valore induet hanc formam

$$\int \frac{p(2pp + m)\partial p}{\sqrt{(\varDelta + \gamma + \varepsilon pp)}} + C\int p\partial p,$$

ubi integratio est manifesta, quod etiam usu venit pro sequentibus formulis litteris $\mathfrak{G}$ etc. affectis. Evidens enim est, si ponatur $\sqrt{(\varDelta + \gamma + \varepsilon pp)} = s$, fieri

$$pp = \frac{ss - \varDelta - \gamma}{\varepsilon} \quad \text{et} \quad p\partial p = \frac{s\partial s}{\varepsilon}$$

ideoque

$$\frac{p\partial p}{\sqrt{(\varDelta + \gamma + \varepsilon pp)}} = \frac{\partial s}{\varepsilon},$$

qua substitutione omnes formulae integrandae fiunt rationales et integrae.

Cum autem iste posterior casus iam satis prolixe sit tractatus ac [...]mplis a rectificatione ellipsis et hyperbolae desumtis illustratus, casus prior, quo tantum erat $\varepsilon = 0$, eo maiore attentione est dignus, quod, quantum equidem scio, a nemine adhuc est observatus, cuius ergo evolutio novae huic methodo unice accepta est referenda. Quemadmodum autem haec deducta sunt ex relatione inter p et q, ita etiam relatio elegantissima erui potest inter has quantitates $p = x + y$ et $u = xy$, quam hic subiungamus.

ANALYSIS PRO INVESTIGANDA RELATIONE INTER p ET u

107. Hic pariter primo in relationem inter ∂p et ∂u inquiramus, et cum sit

$$\frac{\partial p}{\partial u}=\frac{\partial x+\partial y}{y\partial x+x\partial y},$$

ob $\partial x:\partial y=\sqrt{X}:\sqrt{Y}$ erit

$$\frac{\partial p}{\partial u}=\frac{\sqrt{X}+\sqrt{Y}}{y\sqrt{X}+x\sqrt{Y}}$$

et sumtis quadratis

$$\frac{\partial p^2}{\partial u^2}=\frac{X+Y+2\sqrt{XY}}{yyX+xxY+2xy\sqrt{XY}}.$$

Supra autem vidimus esse

$$(\sqrt{X}+\sqrt{Y})^2=qq(\varDelta+\gamma+\delta p+\varepsilon pp)$$

existente $q=y-x$. Pro denominatore autem utamur relatione § 87 inventa

$$\varDelta=\frac{2\alpha+\beta(x+y)+2\gamma xy+\delta xy(x+y)+2\varepsilon xxyy+2\sqrt{XY}}{(y-x)^2},$$

unde fit

$$2\sqrt{XY}=\varDelta qq-2\alpha-\beta p-2\gamma u-\delta pu-2\varepsilon uu,$$

quo valore substituto aequatio nostra erit

$$\frac{\partial p^2}{\partial u^2}=\frac{qq(\varDelta+\gamma+\delta p+\varepsilon pp)}{yyX+xxY+\varDelta qqu-2\alpha u-\beta pu-2\gamma uu-\delta puu-2\varepsilon u^3}.$$

108. Hic autem substitutis loco X et Y valoribus habebimus primo

$$yyX+xxY=\alpha(xx+yy)+\beta xy(x+y)+2\gamma xxyy+\delta xxyy(x+y)$$
$$+\varepsilon xxyy(xx+yy),$$

quae ob $x+y=p$, $xy=u$ et $xx+yy=pp-2u$ erit

$$yyX+xxY=\alpha(pp-2u)+\beta pu+2\gamma uu+\delta puu+\varepsilon uu(pp-2u),$$

unde totus denominator reperietur fore

$$\alpha(pp-4u)+\varepsilon uu(pp-4u)+\varDelta qqu;$$

quare, cum sit $pp-4u=qq$, nostra fractio erit

$$\frac{\partial p^2}{\partial u^2}=\frac{\varDelta+\gamma+\delta p+\varepsilon pp}{\varDelta u+\alpha+\varepsilon uu},$$

unde sequitur haec aequatio separata

$$\frac{\partial p}{\sqrt{(\varDelta+\gamma+\delta p+\varepsilon pp)}}=\frac{\partial u}{\sqrt{(\alpha+\varDelta u+\varepsilon uu)}};$$

unde deducitur hoc

THEOREMA MEMORABILE

109. *Si inter binas variabiles x et y habeatur haec aequatio differentialis*

$$\frac{\partial x}{\sqrt{(\alpha+\beta x+\gamma xx+\delta x^3+\varepsilon x^4)}}=\frac{\partial y}{\sqrt{(\alpha+\beta y+\gamma yy+\delta y^3+\varepsilon y^4)}},$$

tum posito $x+y=p$ et $xy=u$ inter has variabiles p et u semper locum habebit haec aequatio differentialis

$$\frac{\partial p}{\sqrt{(\varDelta+\gamma+\delta p+\varepsilon pp)}}=\frac{\partial u}{\sqrt{(\alpha+\varDelta u+\varepsilon uu)}},$$

ubi $\varDelta$ quidem est constans arbitraria in aequationem posteriorem ingressa, contra vero etiam prior aequatio continet constantem arbitrariam β in altera non occurrentem.

110. Aequationis autem posterioris integratio in promptu est. Si enim utrinque multiplicemus per $\sqrt{\varepsilon}$, integrale per logarithmos ita exprimitur

$$l\Big(p\sqrt{\varepsilon}+\frac{\delta}{2\sqrt{\varepsilon}}+\sqrt{(\varDelta+\gamma+\delta p+\varepsilon pp)}\Big)$$
$$=l\Big(u\sqrt{\varepsilon}+\frac{\varDelta}{2\sqrt{\varepsilon}}+\sqrt{(\alpha+\varDelta u+\varepsilon uu)}\Big)+l\Gamma$$

ideoque integrale ita algebraice exprimetur

$$\varepsilon p+\frac{1}{2}\delta+\sqrt{\varepsilon(\varDelta+\gamma+\delta p+\varepsilon pp)}=\Gamma\Big(\varepsilon u+\frac{1}{2}\varDelta+\sqrt{\varepsilon(\alpha+\varDelta u+\varepsilon uu)}\Big).$$

Ubi constans ista Γ facile definitur ex conditione, quod posito $x=f$ fieri debet $y=g$, hoc est, ut posito $p=f+g$ fiat $u=fg$, quippe ex qua conditione constans prior Δ iam est definita.

111. Quo hinc iam facilius sive p per u sive u per p definiri possit, notetur esse

$$\frac{1}{\varepsilon p+\frac{1}{2}\delta+\sqrt{\varepsilon(\Delta+\gamma+\delta p+\varepsilon pp)}}=\frac{\varepsilon p+\frac{1}{2}\delta-\sqrt{\varepsilon(\Delta+\gamma+\delta p+\varepsilon pp)}}{\frac{1}{4}\delta\delta-\varepsilon(\Delta+\gamma)}$$

et

$$\frac{1}{\varepsilon u+\frac{1}{2}\Delta+\sqrt{\varepsilon(\alpha+\Delta u+\varepsilon uu)}}=\frac{\varepsilon u+\frac{1}{2}\Delta-\sqrt{\varepsilon(\alpha+\Delta u+\varepsilon uu)}}{\frac{1}{4}\Delta\Delta-\alpha\varepsilon}.$$

Hinc igitur per inversionem sequens aequatio resultabit

$$\frac{\varepsilon p+\frac{1}{2}\delta-\sqrt{\varepsilon(\Delta+\gamma+\delta p+\varepsilon pp)}}{\frac{1}{4}\delta\delta-\varepsilon(\Delta+\gamma)}=\frac{1}{\Gamma}\cdot\frac{\varepsilon u+\frac{1}{2}\Delta-\sqrt{\varepsilon(\alpha+\Delta u+\varepsilon uu)}}{\frac{1}{4}\Delta\Delta-\alpha\varepsilon}$$

sive

$$\varepsilon p+\frac{1}{2}\delta-\sqrt{\varepsilon(\Delta+\gamma+\delta p+\varepsilon pp)}$$
$$=\frac{\frac{1}{4}\delta\delta-\varepsilon(\Delta+\gamma)}{\Gamma(\frac{1}{4}\Delta\Delta-\alpha\varepsilon)}\cdot\left(\varepsilon u+\frac{1}{2}\Delta-\sqrt{\varepsilon(\alpha+\Delta u+\varepsilon uu)}\right),$$

ex quibus duabus aequationibus sine alio negotio sive p per u sive u per p exprimi poterit.

112. Hoc igitur modo loco variabilis p pro invenienda quantitate V facile introduci posset variabilis u, si quidem loco formulae

$$\frac{\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}$$

substituatur formula ipsi aequalis

$$\frac{\partial u}{\sqrt{(\alpha+\Delta u+\varepsilon uu)}}.$$

Verum hoc modo casus illi, quibus quantitas V fieri potest algebraica, non tam facile patescent; interim tamen etiam hoc modo certi erimus tam casibus, quibus $\varepsilon=0$, quam, quo $\beta=0$, $\delta=0$ et in serie $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. tantum potestates pares occurrunt, omnes integrationes algebraice succedere

debere. Coronidis loco adhuc aliam relationem inter quantitates p et u investigemus, cuius contemplatio insigne incrementum in integratione aequationum polliceri videtur.

ALIA ANALYSIS PRO INVESTIGATIONE RELATIONIS INTER p ET u

113. Cum sit, ut ante vidimus,

$$\frac{\partial p}{\partial u}=\frac{\sqrt{X}+\sqrt{Y}}{y\sqrt{X}+x\sqrt{Y}},$$

multiplicemus supra et infra per $\sqrt{X}+\sqrt{Y}$, ut numerator evadat

$$(\sqrt{X}+\sqrt{Y})^2=qq(\Delta+\gamma+\delta p+\varepsilon pp);$$

tum autem denominator prodibit

$$yX+xY+(x+y)\sqrt{XY},$$

ubi denominatoris pars rationalis dat

$$\alpha(x+y)+2\beta xy+\gamma xy(x+y)+\delta xy(xx+yy)+\varepsilon xy(x^3+y^3),$$

quae expressio ob $x+y=p$, $y-x=q$ et $xy=u$ abit in

$$\alpha p+2\beta u+\gamma pu+\delta u(pp-2u)+\varepsilon pu(pp-3u).$$

Deinde ante vidimus esse

$$2\sqrt{XY}=\Delta qq-2\alpha-\beta p-2\gamma u-\delta pu-2\varepsilon uu,$$

quod ductum in $\frac{1}{2}p$ et superiori additum praebet

$$\frac{1}{2}\Delta pqq-\frac{1}{2}\beta(pp-4u)+\frac{1}{2}\delta u(pp-4u)+\varepsilon pu(pp-4u),$$

quare denominator ob $pp-4u=qq$ induet hanc formam

$$\frac{1}{2}\Delta pqq-\frac{1}{2}\beta qq+\frac{1}{2}\delta uqq+\varepsilon puqq;$$

hinc aequatio erit

$$\frac{\partial p}{\partial u}=\frac{\Delta+\gamma+\delta p+\varepsilon pp}{\frac{1}{2}\Delta p-\frac{1}{2}\beta+\frac{1}{2}\delta u+\varepsilon pu},$$

unde deducitur

$$\partial p\left(\frac{1}{2}\varDelta p-\frac{1}{2}\beta+\frac{1}{2}\delta u+\varepsilon pu\right)=\partial u(\varDelta+\gamma+\delta p+\varepsilon pp),$$

quae ergo certe est integrabilis; id quod adeo inde patet, quod altera variabilis u nusquam ultra primam dimensionem exsurgit.

114. Verum adhuc alio modo relatio inter p et u investigari potest; scilicet aequatio primo inventa

$$\frac{\partial p}{\partial u}=\frac{\sqrt{X}+\sqrt{Y}}{y\sqrt{X}+x\sqrt{Y}}$$

si supra et infra multiplicetur per $\sqrt{Y}-\sqrt{X}$, dabit

$$\frac{\partial p}{\partial u}=\frac{Y-X}{-yX+xY+\sqrt{XY}(y-x)}.$$

Nunc igitur pro numeratore habebimus

$$\beta q+\gamma pq+\delta q(pp-u)+\varepsilon pq(pp-2u).$$

Pro denominatore vero pars rationalis erit

$$-\alpha q+\gamma qu+\delta pqu+\varepsilon qu(pp-u),$$

pars vero irrationalis

$$\frac{1}{2}\varDelta q^3-\alpha q-\frac{1}{2}\beta pq-\gamma qu-\frac{1}{2}\delta pqu-\varepsilon quu,$$

unde totus denominator conficitur

$$\frac{1}{2}\varDelta q^3-2\alpha q-\frac{1}{2}\beta pq+\frac{1}{2}\delta pqu+\varepsilon qu(pp-2u),$$

unde sequitur haec aequatio differentialis

$$\frac{\partial p}{\partial u}=\frac{\beta+\gamma p+\delta(pp-u)+\varepsilon p(pp-2u)}{\frac{1}{2}\varDelta(pp-4u)-2\alpha-\frac{1}{2}\beta p+\frac{1}{2}\delta pu+\varepsilon u(pp-2u)},$$

quae in ordinem redacta ita se habebit

$$\partial p(\varDelta(pp-4u)-4\alpha-\beta p+\delta pu+2\varepsilon u(pp-2u))$$
$$=2\partial u(\beta+\gamma p+\delta(pp-u)+\varepsilon p(pp-2u)),$$

quae iam ita est comparata, ut nulla via eius integrationem instituendi perspici queat, etiamsi eius integrale revera exhibere queamus.

115. Alio insuper modo relationem inter p et u definire licet, si aequationis

$$\frac{\partial p}{\partial u}=\frac{\sqrt{X}+\sqrt{Y}}{y\sqrt{X}+x\sqrt{Y}}$$

posterius membrum supra et infra multiplicemus per $y\sqrt{X}-x\sqrt{Y}$, ut prodeat

$$\frac{\partial p}{\partial u}=\frac{yX-xY+(y-x)\sqrt{XY}}{yyX-xxY}.$$

Nunc enim denominator evadet

$$\alpha pq+\beta qu-\delta quu-\varepsilon pquu.$$

Pro numeratore autem pars rationalis praebet

$$\alpha q-\gamma qu-\delta pqu-\varepsilon qu(pp-u)$$

et pars irrationalis

$$\frac{1}{2}\varDelta q^3-\alpha q-\frac{1}{2}\beta pq-\gamma qu-\frac{1}{2}\delta pqu-\varepsilon quu;$$

totus igitur numerator erit

$$\frac{1}{2}\varDelta q^3-\frac{1}{2}\beta pq-2\gamma qu-\frac{3}{2}\delta pqu-\varepsilon qupp$$

ideoque

$$\frac{\partial p}{\partial u}=\frac{\frac{1}{2}\varDelta(pp-4u)-\frac{1}{2}\beta p-2\gamma u-\frac{3}{2}\delta pu-\varepsilon ppu}{\alpha p+\beta u-\delta uu-\varepsilon puu}$$

sive

$$\text{-}\delta uu-\varepsilon puu)=\partial u(\varDelta(pp-4u)-\beta p-4\gamma u-3\delta pu-2\varepsilon ppu).$$

s non patet, quomodo multiplicator hanc aequationem inte-
investigari debeat, unde nullum est dubium, quin ista con-
rum ad limites analyseos prolatandos conferre possit.

EXEMPLA QUARUNDAM MEMORABILIUM AEQUATIONUM DIFFERENTIALIUM QUAS ADEO ALGEBRAICE INTEGRARE LICET ETIAMSI NULLA VIA PATEAT VARIABILES A SE INVICEM SEPARANDI

Convent. exhib. die 19 Januarii 1778

Commentatio 714 indicis ENESTROEMIANI
Nova acta academiae scientiarum Petropolitanae 13 (1795/6), 1802, p. 3—13
Summarium ibidem p. 53—54

SUMMARIUM

On conçoit bien, et le nom de l'Auteur en est garant, qu'il n'est point question ici de ces équations difficiles à séparer, dont on peut, pour ainsi dire, deviner les intégrales, ni des intégrales particulieres de pareilles équations. Ce seroit à la vérité un sujet fécond, mais peu utile, que d'imaginer d'équations inséparables dont on pût deviner l'intégrale algébrique complette, ou trouver quelque intégrale particuliere. On sçait, par exemple, que si M, N, P, Q, S et V désignent des fonctions de x et y, et que $\partial V = M\partial x + N\partial y$, l'équation finie $V = 0$ satisfait à l'équation différentielle

$$\partial x(PV + MS) + \partial y(QV + NS) = 0,$$

mais que cette fonction V, qu'il seroit facile de trouver pour chaque équation proposée, n'en seroit qu'une intégrale particuliere.

L'intention de feu M. EULER a été de produire dans ce Mémoire des équations différentielles qui se refusent à toutes les méthodes d'intégration connues, et dont néanmoins on peut donner les intégrales complettes et même algébriques. Il déduit de pareilles équations, avec leurs intégrales algébriques, de l'équation différentielle connue

$$\frac{\partial x}{\sqrt{X}} = \frac{\partial y}{\sqrt{Y}},$$

où

$$X = \alpha + 2\beta x + \gamma xx + 2\delta x^3 + \varepsilon x^4 \quad \text{et} \quad Y = \alpha + 2\beta y + \gamma yy + 2\delta y^3 + \varepsilon y^4,$$

dont l'intégrale complette, qu'on peut encore représenter de différentes autres manieres, e.:

$$\frac{\sqrt{X}+\sqrt{Y}}{x-y} = \sqrt{(2\lambda + \gamma + 2\delta(x+y) + \varepsilon(x+y)^2)},$$

λ étant la constante arbitraire introduite par l'intégration. Cette même équation est don: aussi l'intégrale complette de l'équation différentielle

$$\partial x(\alpha + 2\beta y + \gamma yy + 2\delta y^3 + \varepsilon y^4) + \partial y(\alpha + \beta(x+y) + \gamma xy + \delta xy(x+y) + \varepsilon xxyy) = \lambda \partial y(x-y)^2$$

transformée de $\frac{\partial x}{\sqrt{X}} = \frac{\partial y}{\sqrt{Y}}$, qui donne $\frac{\partial x}{\partial y} = \frac{\sqrt{XY}}{Y}$, et d'où résulte la précédente, en mettant à la place de X et Y leurs valeurs. C'est cette intégration qui sert de fondement à ce Mémoire et qui a fourni à feu M. EULER les exemples qui en font le sujet.

1. Facile quidem est huiusmodi aequationes, quotquot lubuerit, exhibere. quarum integralia assignari queant. Si enim pro V accipiatur quaecunque functio binarum variabilium x et y, ita ut sit

$$\partial V = M\partial x + N\partial y,$$

evidens est huic aequationi differentiali

$$\partial x(PV + MS) + \partial y(QV + NS) = 0$$

semper satisfacere aequationem finitam

$$V = 0.$$

Verum hoc integrale tantum est particulare. Praeterea vero si eiusmodi aequatio proponatur, plerumque haud difficulter ista functio V vel divinando inveniri potest, ita ut huiusmodi aequationes parum in recessu habere sint censendae. Hic autem tales aequationes in medium sum allaturus, quarum integratio omnes methodos adhuc cognitas respuere videatur, cum tamen nihilominus earum integralia completa atque adeo algebraica exhiberi queant.

2. Huiusmodi scilicet aequationes differentiales deducere licet ex hac aequatione differentiali hactenus plurimum tractata

$$\frac{\partial x}{\sqrt{X}} = \frac{\partial y}{\sqrt{Y}},$$

in qua est

$$X = \alpha + 2\beta x + \gamma xx + 2\delta x^3 + \varepsilon x^4 \quad \text{et} \quad Y = \alpha + 2\beta y + \gamma yy + 2\delta y^3 + \varepsilon y^4,$$

cuius integrale completum hac aequatione finita exprimitur[1])

$$\text{I.} \quad \frac{\sqrt{X} + \sqrt{Y}}{x - y} = \sqrt{\left(2\lambda + \gamma + 2\delta(x + y) + \varepsilon(x + y)^2\right)},$$

ubi λ denotat constantem arbitrariam integratione ingressam, quod ergo integrale etiam hoc modo exhiberi potest

$$\text{II.} \quad \sqrt{XY} = \lambda(x - y)^2 - \alpha - \beta(x + y) - \gamma xy - \delta xy(x + y) - \varepsilon xxyy.$$

Quin etiam irrationalitatem penitus tollendo hoc integrale sequentem induet formam

$$\text{III.} \quad 0 = \lambda\lambda(x - y)^2 - 2\lambda\left(\alpha + \beta(x + y) + \gamma xy + \delta xy\,(x + y) + \varepsilon xxyy\right)$$
$$+ (\beta\beta - \alpha\gamma) - 2\alpha\delta(x + y) - \alpha\varepsilon(x + y)^2 - 2\beta\delta xy - 2\beta\varepsilon xy(x + y) + (\delta\delta - \gamma\varepsilon)xxyy.$$

Hinc iam sequentia exempla evolvamus.

EXEMPLUM 1

3. Cum ex aequatione $\frac{\partial x}{\sqrt{X}} = \frac{\partial y}{\sqrt{Y}}$ sit $\frac{\partial x}{\partial y} = \sqrt{\frac{X}{Y}}$, habebimus

$$\frac{\partial x}{\partial y} = \frac{\sqrt{XY}}{Y};$$

ubi si valores pro Y et $\sqrt{XY}$ ex forma integralis secunda substituamus, prodibit

$$\frac{\partial x}{\partial y} = \frac{\lambda(x - y)^2 - \alpha - \beta(x + y) - \gamma xy - \delta xy(x + y) - \varepsilon xxyy}{\alpha + 2\beta y + \gamma yy + 2\delta y^3 + \varepsilon y^4},$$

quae more solito in ordinem redacta hanc induet formam

$$\partial x(\alpha + 2\beta y + \gamma yy + 2\delta y^3 + \varepsilon y^4) + \partial y(\alpha + \beta(x + y) + \gamma xy + \delta xy(x + y) + \varepsilon xxyy)$$
$$= \lambda \partial y(x - y)^2,$$

1) Vide p. 18 et 211. A. K.

cuius aequationis ergo integrale est aequatio finita, quam sub triplici forma exhibuimus. Quoniam autem in hoc integrali nulla nova constans occurrit, quae in differentiali non insit, hoc integrale tantum pro particulari est habendum.

4. Interim tamen haec aequatio differentialis iam ita est comparata, ut nemo certe eius integrale divinando elicere potuerit, cum sex quantitates diversae ibi occurrant. Quin etiam si quatuor adeo litterae evanescant, tamen integrale adhuc satis absconditum deprehenditur. Veluti si sumamus

$$\beta = \gamma = \delta = \varepsilon = 0,$$

oritur haec aequatio differentialis

$$\alpha \partial x + \alpha \partial y = \lambda \partial y (x - y)^2,$$

cuius ergo integrale ex prima forma erit

$$\frac{2\sqrt{\alpha}}{x-y} = \sqrt{2\lambda} \quad \text{sive} \quad x - y = \frac{\sqrt{2\alpha}}{\lambda} \quad \text{sive} \quad x = y + \frac{\sqrt{2\alpha}}{\lambda},$$

qui valor utique satisfacit, sed tantum particulariter. Pro integrali autem completo inveniendo statuatur $x - y = v$ sive $x = y + v$, unde aequatio differentialis evadet

$$\partial y = \frac{\alpha \partial v}{\lambda vv - 2\alpha},$$

cuius ergo integrale completum sive a logarithmis sive ab arcubus circularibus pendet.

5. Ponamus nunc esse $\alpha = \gamma = \delta = \varepsilon = 0$ et aequatio nostra differentialis erit

$$2\beta y \partial x + \beta(x + y)\partial y = \lambda \partial y (x - y)^2,$$

cui ergo satisfacit hoc integrale ex I forma

$$\frac{\sqrt{2\beta x} + \sqrt{2\beta y}}{x-y} = \sqrt{2\lambda}$$

vel ex II forma

$$2\beta \sqrt{xy} = \lambda (x - y)^2 - \beta (x + y).$$

Illa autem forma praebet

$$\sqrt{x} + \sqrt{y} = (x - y)\sqrt{\frac{\lambda}{\beta}},$$

quae divisa per $\sqrt{x} + \sqrt{y}$ dat

$$1 = (\sqrt{x} - \sqrt{y})\sqrt{\frac{\lambda}{\beta}} \quad \text{sive} \quad \sqrt{x} = \sqrt{y} + \sqrt{\frac{\beta}{\lambda}}$$

hincque

$$x = y + 2\sqrt{\frac{\beta}{\lambda}y} + \frac{\beta}{\lambda}$$

ideoque

$$\partial x = \partial y + \frac{\partial y \sqrt{\beta}}{\sqrt{\lambda y}},$$

qui valores substituti aequationem identicam producunt.

6. Cum igitur isti casus simplicissimi iam profundiorem indagationem requirant, hinc evidentissime elucet, si omnes sex litterae in calculo relinquantur, tum neminem certe unquam eius integrale saltem particulare esse eruturum; unde haec ipsa aequatio generalis

$$\partial x(\alpha + 2\beta y + \gamma yy + 2\delta y^3 + \varepsilon y^4) + \partial y(\alpha + \beta(x + y) + \gamma xy + \delta xy(x + y) + \varepsilon xxyy) \\ = \lambda \partial y(x - y)^2$$

omni attentione maxime digna videtur, cum eius integrale, licet particulare, sit ipsa aequatio supra § 2 assignata sub triplici forma. In sequentibus autem exemplis huiusmodi aequationes differentiales proferemus, quarum adeo integralia completa algebraice exhiberi queant.

EXEMPLUM 2

7. Cum sit $\partial x : \partial y = \sqrt{X} : \sqrt{Y}$, erit

$$\frac{\partial x + \partial y}{\partial x - \partial y} = \frac{\sqrt{X} + \sqrt{Y}}{\sqrt{X} - \sqrt{Y}}.$$

Iam haec fractio supra et infra multiplicetur per $\sqrt{X} + \sqrt{Y}$ fietque

$$\frac{\partial x + \partial y}{\partial x - \partial y} = \frac{(\sqrt{X} + \sqrt{Y})^2}{X - Y},$$

cuius numerator ex prima forma integralis est

$$(x-y)^2(2\lambda+\gamma+2\delta(x+y)+\varepsilon(x+y)^2);$$

denominator vero erit

$$2\beta(x-y)+\gamma(xx-yy)+2\delta(x^3-y^3)+\varepsilon(x^4-y^4)$$

sicque haec fractio per $x-y$ deprimi potest, ita ut habeamus

$$\frac{\partial x+\partial y}{\partial x-\partial y}=\frac{(x-y)(2\lambda+\gamma+2\delta(x+y)+\varepsilon(x+y)^2)}{2\beta+\gamma(x+y)+2\delta(xx+xy+yy)+\varepsilon(x+y)(xx+yy)},$$

cuius ergo integrale pariter erit ipsa aequatio finita supra assignata; quae cum praeter quantitates constantes in ipsam aequationem differentialem ingredientes, quae sunt β, γ, δ, ε et λ, insuper litteram α contineat, utique pro integrali completo est habenda.

8. Quo hanc aequationem in ordinem redigamus, primo eam in hanc formam convertamus

$$\frac{\partial x}{\partial y}=\frac{\beta+\lambda(x-y)+\gamma x+\delta x(2x+y)+\varepsilon xx(x+y)}{\lambda(x-y)-\beta-\gamma y-\delta y(2y+x)-\varepsilon yy(x+y)}.$$

Nunc igitur fractionibus sublatis prodibit haec aequatio

$$\left.\begin{aligned}&\lambda\partial x(x-y)-\beta\partial x-\gamma y\partial x-\delta y\partial x(2y+x)-\varepsilon yy\partial x(x+y)\\ -&\lambda\partial y(x-y)-\beta\partial y-\gamma x\partial y-\delta x\partial y(2x+y)-\varepsilon xx\partial y(x+y)\end{aligned}\right\}=0.$$

Huius ergo aequationis integrale completum est ipsa illa aequatio finita, quam supra sub triplici forma repraesentavimus, in qua littera α est constans arbitraria per integrationem ingressa, unde ex tertia forma integrale ita referri poterit

$$\alpha(2\lambda+\gamma+2\delta(x+y)+\varepsilon(x+y)^2)=\lambda\lambda(x-y)^2-2\lambda\beta(x+y)-2\lambda\gamma xy$$
$$-2\lambda\delta xy(x+y)-2\lambda\varepsilon xxyy+\beta\beta-2\beta\delta xy-2\beta\varepsilon xy(x+y)+(\delta\delta-\gamma\varepsilon)xxyy$$

sive

$$\alpha=\left\{\begin{aligned}&\frac{\lambda\lambda(x-y)^2-2\lambda\beta(x+y)-2\lambda\gamma xy-2\lambda\delta xy(x+y)-2\lambda\varepsilon xxyy}{2\lambda+\gamma+2\delta(x+y)+\varepsilon(x+y)^2}\\ &\frac{+\beta\beta-2\beta\delta xy-2\beta\varepsilon xy(x+y)+(\delta\delta-\gamma\varepsilon)xxyy}{2\lambda+\gamma+2\delta(x+y)+\varepsilon(x+y)^2}\end{aligned}\right\}$$

9. Quia in hac aequatione plures occurrunt litterae, scilicet λ, β, γ, δ, ε, contemplemur primo casus speciales, quibus duae tantum litterae occurrunt reliquis ad nihilum redactis.

CASUS 1

QUO $\gamma = \delta = \varepsilon = 0$

10. Aequatio ergo differentialis erit

$$\lambda \partial x(x-y) - \lambda \partial y(x-y) - \beta \partial x - \beta \partial y = 0$$

sive

$$\lambda(x-y)(\partial x - \partial y) - \beta(\partial x + \partial y) = 0,$$

cuius integrale sponte prodit

$$\lambda(x-y)^2 - 2\beta(x+y) = \text{Const.}$$

Generalis vero integralis forma hoc casu praebet

$$\alpha = \frac{\lambda\lambda(x-y)^2 - 2\lambda\beta(x+y) + \beta\beta}{2\lambda}.$$

CASUS 2

QUO $\beta = \delta = \varepsilon = 0$

Hoc casu aequatio differentialis erit

$$\lambda \partial x(x-y) - \lambda \partial y(x-y) - \gamma(y \partial x + x \partial y) = 0,$$

cuius integrale pariter sponte se offert, quandoquidem erit

$$\lambda(x-y)^2 - 2\gamma xy = \text{const.}$$

Ex forma generali integrale fit

$$\alpha = \frac{\lambda\lambda(x-y)^2 - 2\lambda\gamma xy}{2\lambda + \gamma}.$$

Quin etiam si fuerit tantum $\delta = \varepsilon = 0$, qui sit

CASUS 3

aequatio differentialis erit

$$\lambda(x-y)(\partial x - \partial y) - \beta(\partial x + \partial y) - \gamma(y \partial x + x \partial y) = 0,$$

cuius integrale est manifesto

$$\lambda(x-y)^2-2\beta(x+y)-2\gamma xy=\text{const.}$$

Forma generalis autem praebet

$$\alpha=\frac{\lambda\lambda(x-y)^2-2\lambda\beta(x+y)-2\lambda\gamma xy+\beta\beta}{2\lambda+\gamma},$$

ubi consensus est manifestus, sicque, quoties ambae litterae δ et ε evanescunt, res nihil plane habet in recessu; verum si litterarum δ et ε vel altera tantum vel ambae affuerint, eiusmodi oriuntur aequationes differentiales, quarum integratio per methodos usitatas non parum difficultatis involvit; huiusmodi igitur casus hic data opera evolvamus.

CASUS 4
QUO $\beta=\gamma=\varepsilon=0$

11. Hoc ergo casu aequatio differentialis erit

$$\lambda(x-y)(\partial x-\partial y)-\delta y\partial x(2y+x)-\delta x\partial y(2x+y)=0,$$

cuius integrale ex forma generali resultat

$$\alpha=\frac{\lambda\lambda(x-y)^2-2\lambda\delta xy(x+y)+\delta\delta xxyy}{2\lambda+2\delta(x+y)},$$

cuius veritas neutiquam tam clare perspicitur quam casibus praecedentibus; namque posito brevitatis gratia $\lambda=n\delta$, ut habeatur haec aequatio

$$n(x-y)(\partial x-\partial y)=y\partial x(2y+x)+x\partial y(2x+y);$$

eius prius membrum sponte est integrabile hincque etiam, si multiplicetur per functionem quamcunque $x-y$. Verum nulla huiusmodi functio datur, qua etiam posterius membrum integrabile reddatur. Ut autem more solito in eius integrale inquiramus, ponamus

$$x+y=p \quad\text{et}\quad x-y=q,$$

ut sit

$$x=\frac{p+q}{2} \quad\text{et}\quad y=\frac{p-q}{2},$$

atque aequatio nostra induet hanc formam

$$nq\partial q = \frac{1}{4}\partial p(3pp + qq) - pq\partial q.$$

Ponamus hic $qq = v$, ut sit $2q\partial q = \partial v$, et aequatio nostra erit

$$2n\partial v + 2p\partial v - v\partial p = 3pp\partial p.$$

In qua aequatione quia v unicam tantum habet dimensionem, ea methodo consueta resolvi poterit; divisa enim per $2n + 2p$ praebet

$$\partial v - \frac{v\partial p}{2n + 2p} = \frac{3pp\partial p}{2n + 2p}.$$

12. Constat autem hanc aequationem generalem $\partial v + Pv\partial p = Q\partial p$, ubi P et Q sint functiones quaecunque ipsius p, integrabilem reddi, si ducatur in $e^{\int P\partial p}$; tum enim integrale fit

$$e^{\int P\partial p} v = \int e^{\int P\partial p} Q\partial p.$$

Hinc autem pro nostro casu habebimus

$$P = \frac{-1}{2n + 2p} \quad \text{et} \quad Q = \frac{3pp}{2n + 2p};$$

quamobrem fiet

$$\int P\partial p = -\frac{1}{2}l(2n + 2p) + \frac{1}{2}l2 = -\frac{1}{2}l(n + p)$$

ideoque

$$e^{\int P\partial p} = \frac{1}{\sqrt{(n + p)}};$$

ergo aequatio integralis erit

$$\frac{v}{\sqrt{(n + p)}} = \frac{3}{2}\int \frac{pp\partial p}{(n + p)^{\frac{3}{2}}}.$$

Pro postremo membro ponatur

$$n + p = zz \quad \text{sive} \quad p = zz - n$$

eritque

$$(n + p)^{\frac{3}{2}} = z^3;$$

tum vero fiet

$$\frac{pp\partial p}{(n+p)^{\frac{3}{2}}}=\frac{2\partial z(z^4-2nzz+nn)}{zz}=2zz\partial z-4n\partial z+\frac{2nn\partial z}{zz},$$

cuius integrale est $\frac{2}{3}z^3-4nz-\frac{2nn}{z}$; consequenter nostra aequatio integralis erit

$$\frac{v}{\sqrt{(n+p)}}=z^3-6nz-\frac{3nn}{z}+\text{const.}$$

sive

$$\frac{v}{\sqrt{(n+p)}}=(n+p)^{\frac{3}{2}}-6n\sqrt{(n+p)}-\frac{3nn}{\sqrt{(n+p)}}+C,$$

quae aequatio reducitur ad hanc formam

$$v=(n+p)^2-6n(n+p)-3nn+C\sqrt{(n+p)}$$

sive

$$v=pp-4np-8nn+C\sqrt{(n+p)}.$$

13. Erat autem $v=qq$ sicque integrale nostrum erit

$$qq=pp-4np-8nn+C\sqrt{(n+p)}.$$

At vero integrale supra datum si pariter ad quantitates p et q reducatur, in hanc formam transmutatur

$$\frac{2\alpha}{\delta}=\frac{nnqq-\frac{np(pp-qq)}{2}+\frac{(pp-qq)^2}{16}}{n+p}=\frac{16nnqq-8np(pp-qq)+(pp-qq)^2}{16(n+p)}.$$

Ex forma autem inventa constans arbitraria C hoc modo definitur

$$-C=\frac{pp-qq-4np-8nn}{\sqrt{(n+p)}},$$

cuius quadratum praebet

$$CC=\frac{(pp-qq)^2-8np(pp-qq)-16nn(pp-qq)+16nnpp+64n^3p+64n^4}{n+p},$$

hincque iam elicitur $\frac{32\alpha}{\delta}-CC=64n^3$. Unde patet ambo haec integralia perfecte inter se convenire, siquidem tantum quantitate constante a se invicem discrepant.

14. Ob tantas ergo ambages, quibus usi sumus ad integrale eliciendum, iste casus tanto maiore attentione dignus est censendus. Interim tamen, quoniam integrale denominatorem habet $n+p$ atque ipsa fractio differentiata nostram aequationem differentialem reproducere debet, necesse est, ut ipsa nostra aequatio differentialis

$$4nq\partial q + 4pq\partial q - 3pp\partial p - qq\partial p = 0$$

integrabilis reddatur, si per certam fractionem, quae reperitur $\frac{pp-qq+4np+8nn}{(n+p)^2}$, multiplicetur, id quod calculum instituenti per plures demum ambages patebit, si formulam pro $\frac{32\alpha}{\delta}$ supra exhibitam differentiare voluerit, quem laborem autem hic suscipere non vacat, praesertim postquam consensum amborum integralium iam ostenderimus; quam ob causam iste casus maximam attentionem meretur.

CASUS 5

QUO $\beta = \gamma = \delta = 0$

15. Hoc ergo casu aequatio differentialis erit

$$\lambda(x-y)(\partial x - \partial y) - \varepsilon(x+y)(yy\partial x + xx\partial y) = 0;$$

cuius ergo integrale completum erit

$$\alpha = \frac{\lambda\lambda(x-y)^2 - 2\lambda\varepsilon xxyy}{2\lambda + \varepsilon(x+y)^2}.$$

Fiat nunc iterum $x+y=p$ et $x-y=q$ ponaturque $\lambda = n\varepsilon$ et aequatio differentialis prodibit

$$nq\partial q - \frac{1}{4}p\partial p(pp+qq) + \frac{1}{2}ppq\partial q = 0.$$

Integrale vero erit

$$\frac{\alpha}{\varepsilon} = \frac{nnqq - \frac{1}{8}n(pp-qq)^2}{2n+pp}.$$

Ista autem aequatio pariter nulla laborat difficultate; posito enim $qq = v$, ut sit $2q\partial q = \partial v$, prodibit haec forma

$$2n\partial v - pv\partial p + pp\partial v = p^3\partial p$$

hacque divisa per $2n+pp$ erit

$$\partial v - \frac{vp\partial p}{2n+pp} = \frac{p^3\partial p}{2n+pp},$$

quae cum aequatione generali § 12 comparata dat

$$P = \frac{-p}{2n+pp} \quad \text{et} \quad Q = \frac{p^3}{2n+pp}.$$

Fiet ergo

$$\int P\partial p = -\frac{1}{2}l(2n+pp)$$

ideoque

$$e^{\int P\partial p} = \frac{1}{\sqrt{(2n+pp)}};$$

ergo aequatio integralis erit

$$\frac{v}{\sqrt{(2n+pp)}} = \int \frac{p^3\partial p}{(2n+pp)^{\frac{3}{2}}} = \frac{4n+pp}{\sqrt{(2n+pp)}} + \text{Const.}$$

Sicque integrale completum erit

$$qq = 4n + pp + C\sqrt{(2n+pp)}$$

sive habebimus

$$C = \frac{qq-pp-4n}{\sqrt{(2n+pp)}};$$

quae forma, ut cum supra assignata comparari possit, quadretur fietque

$$CC = \frac{q^4 - 2ppqq - 8nqq + p^4 + 8npp + 16nn}{2n+pp}.$$

Erat autem

$$-\frac{8\alpha}{n\varepsilon} = \frac{(pp-qq)^2 - 8nqq}{2n+pp},$$

quarum expressionum differentia est $CC + \frac{8\alpha}{n\varepsilon} = 8n$; unde patet constantem C ita definiri, ut sit $CC = 8n - \frac{8\alpha}{n\varepsilon}$.

CASUS GENERALIS
UBI OMNES LITTERAE ADMITTUNTUR

16. Posito nunc in genere $x+y=p$ et $x-y=q$ aequatio nostra differentialis erit

$$\beta\partial p - \frac{1}{2}\gamma(p\partial p - q\partial q) - \frac{1}{4}\delta\partial p(3pp+qq)$$
$$-\beta\partial q - \frac{1}{4}\varepsilon p\partial p(pp+qq) + \frac{1}{2}\varepsilon ppq\partial q = 0,$$

cuius ergo integrale completum erit

$$a = \left\{ \begin{array}{l} \dfrac{+\lambda\lambda qq - 2\lambda\beta p - \frac{1}{2}\lambda\gamma(pp-qq) - \frac{1}{2}\lambda\delta p(pp-qq) - \frac{1}{8}\lambda\varepsilon(pp-qq)^2}{2\lambda+\gamma+2\delta p+\varepsilon pp} \\ \dfrac{+\beta\beta - \frac{1}{2}\beta\delta(pp-qq) - \frac{1}{2}\beta\varepsilon p(pp-qq) + \frac{1}{16}(\delta\delta-\gamma\varepsilon)(pp-qq)^2}{2\lambda+\gamma+2\delta p+\varepsilon pp} \end{array} \right\}$$

17. Postquam autem nostra aequatio ad hanc formam est reducta, eius resolutio nulla amplius difficultate laborat; posito enim $qq = v$ et terminis sive v sive ∂v continentibus in unam partem translatis ista forma proveniet

$$(2\lambda+\gamma+2\delta p+\varepsilon pp)\partial v - v(\delta+\varepsilon p)\partial p = (4\beta+2\gamma p+3\delta pp+\varepsilon p^3)\partial p$$

sive

$$\partial v - \frac{v\partial p(\delta+\varepsilon p)}{2\lambda+\gamma+2\delta p+\varepsilon pp} = \frac{\partial p(4\beta+2\gamma p+3\delta pp+\varepsilon p^3)}{2\lambda+\gamma+2\delta p+\varepsilon pp};$$

haec forma cum generali (§ 12) comparata dat

$$P = \frac{-\delta-\varepsilon p}{2\lambda+\gamma+2\delta p+\varepsilon pp} \quad \text{et} \quad Q = \frac{4\beta+2\gamma p+3\delta pp+\varepsilon p^3}{2\lambda+\gamma+2\delta p+\varepsilon pp};$$

fiet ergo

$$\int P\partial p = -\frac{1}{2}l(2\lambda+\gamma+2\delta p+\varepsilon pp)$$

ideoque

$$e^{\int P\partial p} = \frac{1}{\sqrt{(2\lambda+\gamma+2\delta p+\varepsilon pp)}},$$

quocirca integratio dabit

$$\frac{v}{\sqrt{(2\lambda+\gamma+2\delta p+\varepsilon pp)}} = \int \frac{\partial p(4\beta+2\gamma p+3\delta pp+\varepsilon p^3)}{(2\lambda+\gamma+2\delta p+\varepsilon pp)^{\frac{3}{2}}}.$$

18. Ut nunc postremam formulam integralem facillime evolvamus, ponamus eius integrale esse

$$\frac{A+Bp+Cpp}{\sqrt{(2\lambda+\gamma+2\delta p+\varepsilon pp)}},$$

cuius formae differentiale debitum habebit denominatorem, at vero numerator ad hanc formam reducitur

$$\partial p((2\lambda+\gamma)B - A\delta) + p\partial p(B\delta + 2C(2\lambda+\gamma) - A\varepsilon) + pp\partial p\cdot 3\delta C + p^3\partial p\cdot\varepsilon C;$$

hinc ergo obtinemus quatuor sequentes aequationes

$$1.\quad 4\beta = (2\lambda + \gamma)B - A\delta,$$

$$2.\quad 2\gamma = B\delta + 2C(2\lambda + \gamma) - A\varepsilon,$$

$$3.\quad 3\delta = 3\delta C,$$

$$4.\quad \varepsilon = \varepsilon C,$$

ubi binae postremae manifesto praebent $C = 1$; tum vero secunda fit

$$B\delta + 4\lambda - A\varepsilon = 0,$$

ex qua cum prima coniuncta elicitur

$$B = \frac{4\beta\varepsilon + 4\lambda\delta}{(2\lambda + \gamma)\varepsilon - \delta\delta}$$

ac denique

$$A = \frac{4\beta\delta + 4\lambda(2\lambda + \gamma)}{(2\lambda + \gamma)\varepsilon - \delta\delta};$$

quibus valoribus inventis aequatio nostra integralis erit

$$\frac{qq}{\sqrt{(2\lambda + \gamma + 2\delta p + \varepsilon pp)}} = \frac{A + Bp + Cpp}{\sqrt{(2\lambda + \gamma + 2\delta p + \varepsilon pp)}} + \varDelta$$

sive

$$\varDelta = \frac{qq - A - Bp - Cpp}{\sqrt{(2\lambda + \gamma + 2\delta p + \varepsilon pp)}} \quad \text{sive} \quad -\varDelta = \frac{Cpp - qq + A + Bp}{\sqrt{(2\lambda + \gamma + 2\delta p + \varepsilon pp)}},$$

cuius quadratum a valore ipsius $\frac{16\alpha}{\delta\delta - \gamma\varepsilon - 2\lambda\varepsilon}$ subtractum relinquit quantitatem constantem.

DE INFINITIS CURVIS ALGEBRAICIS QUARUM LONGITUDO INDEFINITA ARCUI ELLIPTICO AEQUATUR

Convent. exhib. die 20 Augusti 1781

Commentatio 780 indicis ENESTROEMIANI

Mémoires de l'académie des sciences de St. Pétersbourg 11, 1830, p. 95—99

1. Proposueram ante aliquot annos[1]) duo theoremata, quae mihi quidem omni attentione digna videbantur, quorum altero statui *nullam prorsus dari curvam algebraicam, cuius longitudo indefinita cuipiam logarithmo aequatur;* altero vero negavi *praeter circulum ullam exhiberi posse curvam algebraicam, cuius longitudo indefinita arcui cuipiam circulari aequatur.* Utrum vero aliae dentur lineae curvae, quarum rectificatio ita ipsis sit propria, ut eadem nullis aliis curvis algebraicis conveniat, quaestio est maxime ardua.

2. Inveni quidem nonnullas curvas algebraicas, quarum longitudo indefinita aequatur arcui elliptico atque adeo etiam parabolico, at vero nullam adhuc investigare mihi licuit eiusmodi curvam algebraicam, cuius rectificatio cum hyperbola conveniret. Nuper autem incidi in eiusmodi formulas, quae infinitas praebent curvas algebraicas, quarum omnium longitudo indefinita ad arcum ellipticum reduci potest, quas idcirco curvas hic in medium attulisse operae pretium videtur, siquidem hoc argumentum plane est novum neque a quoquam satis dilucide pertractatum.

---

1) Vide p. 88 et 83. A. K.

3. Consideravi scilicet curvam, cuius coordinatae orthogonales x et y his formulis exprimantur

$$x = \frac{a\cos.(n+1)\varphi}{n+1} + \frac{b\cos.(n-1)\varphi}{n-1},$$

$$y = \frac{a\sin.(n+1)\varphi}{n+1} + \frac{b\sin.(n-1)\varphi}{n-1}.$$

Hinc ergo erit

$$\frac{\partial x}{\partial \varphi} = -a\sin.(n+1)\varphi - b\sin.(n-1)\varphi,$$

$$\frac{\partial y}{\partial \varphi} = \quad a\cos.(n+1)\varphi + b\cos.(n-1)\varphi.$$

Hinc ergo erit elementum curvae

$$\sqrt{\partial x^2 + \partial y^2} = \partial\varphi\sqrt{aa + bb + 2ab\cos.2\varphi},$$

quae formula manifesto rectificationem ellipsis involvit. Nam si coordinatae statuantur in ellipsi

$$X = f\cos.\varphi \quad \text{et} \quad Y = g\sin.\varphi,$$

erit

$$\sqrt{\partial X^2 + \partial Y^2} = \partial\varphi\sqrt{ff\sin.\varphi^2 + gg\cos.\varphi^2},$$

quae formula ob

$$\sin.\varphi^2 = \frac{1-\cos.2\varphi}{2} \quad \text{et} \quad \cos.\varphi^2 = \frac{1+\cos.2\varphi}{2}$$

abit in hanc

$$\partial\varphi\sqrt{\frac{ff+gg}{2} + \frac{gg-ff}{2}\cos.2\varphi},$$

ubi si sumamus $g = a+b$ et $f = a-b$, ipsa nostra formula resultat, ita ut ellipseos eandem rectificationem habentis sint semiaxes $a+b$ et $a-b$.

4. Quoniam igitur in elemento curvae $\sqrt{\partial x^2 + \partial y^2}$ numerus n non inest ideoque arbitrio nostro prorsus relinquitur, manifestum est innumerabiles exhiberi posse curvas algebraicas, quarum arcus adeo datae ellipseos arcubus aequentur, quae omnes curvae inter se maxime erunt diversae atque pro variis valoribus loco n assumtis ad ordines curvarum algebraicarum plurimum diversos erunt referendae. Neque tamen hinc sequitur, etiamsi circulus sit species ellipsis, pro circulo quoque alias diversas curvas eiusdem rectificationis hoc modo assignari posse. Cum enim circulus prodeat, si ambo semiaxes f

et g statuantur aequales, necesse est, ut vel a vel b evanescat. Sumto autem $b = 0$ erit

$$x = \frac{a \cos. (n+1)\varphi}{n+1} \quad \text{et} \quad y = \frac{a \sin. (n+1)\varphi}{n+1}$$

sicque erit $xx + yy = \frac{aa}{(n+1)^2}$; quicquid pro n accipiatur, semper igitur circulus oritur.

5. Cum autem casu in istas formulas tantum incidissem, utique operae pretium erit in eiusmodi Analysin inquirere, quae proposita ellipsi via directa ad formulas supra § 3 allatas manuducat, quem in finem sequens problema resolvendum suscipio.

PROBLEMA

6. *Proposita ellipsi, cuius coordinatae orthogonales X et Y his formulis definiantur*

$$X = 2f \cos. \theta \quad et \quad Y = 2g \sin. \theta,$$

invenire innumerabiles alias curvas algebraicas, quae cum ista ellipsi communem rectificationem sortiantur.

SOLUTIO

Sint x et y coordinatae curvarum quaesitarum, et cum esse oporteat $\partial x^2 + \partial y^2 = \partial X^2 + \partial Y^2$, haec conditio implebitur, si sumatur

$$\begin{aligned}\partial x &= \partial X \cos. \varphi + \partial Y \sin. \varphi, \\ \partial y &= \partial X \sin. \varphi - \partial Y \cos. \varphi.\end{aligned}$$

Iam quia hae formulae differentiales integrationem admittere debent, integrentur, qua fieri licet, more solito ac reperietur

$$\begin{aligned}x &= X \cos. \varphi + Y \sin. \varphi + \int \partial \varphi (X \sin. \varphi - Y \cos. \varphi), \\ y &= X \sin. \varphi - Y \cos. \varphi - \int \partial \varphi (X \cos. \varphi + Y \sin. \varphi).\end{aligned}$$

7. Cum iam sit $X = 2f \cos. \theta$ et $Y = 2g \sin. \theta$, sumamus angulum $\varphi = n\theta$ eritque per notas angulorum reductiones

$$
\begin{aligned}
X \sin.\varphi &= \quad f \sin.(n+1)\theta + f \sin.(n-1)\theta,\\
X \cos.\varphi &= \quad f \cos.(n+1)\theta + f \cos.(n-1)\theta,\\
Y \sin.\varphi &= -g \cos.(n+1)\theta + g \cos.(n-1)\theta,\\
Y \cos.\varphi &= \quad g \sin.(n+1)\theta - g \sin.(n-1)\theta.
\end{aligned}
$$

Ex his iam valoribus colligitur

$$
\begin{aligned}
X \sin.\varphi - Y \cos.\varphi &= (f-g)\sin.(n+1)\theta + (f+g)\sin.(n-1)\theta,\\
X \cos.\varphi + Y \sin.\varphi &= (f-g)\cos.(n+1)\theta + (f+g)\cos.(n-1)\theta,
\end{aligned}
$$

quae aequationes ductae in $\partial\varphi = n\partial\theta$ et integratae, si brevitatis gratia ponatur $f+g=b$ et $f-g=a$, dabunt

$$
\begin{aligned}
\int \partial\varphi (X \sin.\varphi - Y \cos.\varphi) &= -\frac{na \cos.(n+1)\theta}{n+1} - \frac{nb \cos.(n-1)\theta}{n-1},\\
\int \partial\varphi (X \cos.\varphi + Y \sin.\varphi) &= +\frac{na \sin.(n+1)\theta}{n+1} + \frac{nb \sin.(n-1)\theta}{n-1}.
\end{aligned}
$$

8. Si igitur pro integralibus hi valores substituantur, nostrae coordinatae erunt

$$
\begin{aligned}
x &= a \cos.(n+1)\theta + b \cos.(n-1)\theta - \frac{na}{n+1}\cos.(n+1)\theta - \frac{nb}{n-1}\cos.(n-1)\theta,\\
y &= a \sin.(n+1)\theta + b \sin.(n-1)\theta - \frac{na}{n+1}\sin.(n+1)\theta - \frac{nb}{n-1}\sin.(n-1)\theta.
\end{aligned}
$$

At binis membris rite coniunctis istae coordinatae pro curvis quaesitis cum ellipsi communem rectificationem habentibus ita erunt expressae

$$
\begin{aligned}
x &= \frac{a}{n+1}\cos.(n+1)\theta - \frac{b}{n-1}\cos.(n-1)\theta,\\
y &= \frac{a}{n+1}\sin.(n+1)\theta - \frac{b}{n-1}\sin.(n-1)\theta,
\end{aligned}
$$

quae expressiones a supra allatis aliter non differunt, nisi quod hic littera b negative sit sumta. Ubi notandum casu, quo $n=0$, ipsam ellipsin esse prodituram. Posito enim $n=0$ fiet

$$x = (a+b)\cos.\theta \quad \text{et} \quad y = (a-b)\sin.\theta.$$

9. Si sumatur $n=2$, prodibit sine dubio curva post ellipsin simplicissima. Reperietur autem

$$x=\frac{a}{3}\cos.3\theta-b\cos.\theta \quad \text{et} \quad y=\frac{a}{3}\sin 3\theta-b\sin\theta.$$

Loco $\frac{a}{3}$ scribamus litteram c et quaeramus chordam $\sqrt{xx+yy}=z$ oritque $zz=cc+bb-2bc\cos.2\theta$, consequenter

$$\cos.2\theta=\frac{bb+cc-zz}{2bc}$$

hincque

$$\sin.\theta=\sqrt{\frac{zz-(b-c)^2}{4bc}} \quad \text{et} \quad \cos.\theta=\sqrt{\frac{(b+c)^2-zz}{4bc}}.$$

Hinc, cum sit $\sin.3\theta=4\sin.\theta\cos.\theta^2-\sin.\theta$ et $\cos.3\theta=4\cos.\theta^3-3\cos.\theta$, si angulus θ eliminetur, eruetur aequatio inter ipsas coordinatas x et y, quae autem ad plures dimensiones assurget.

10. Methodus, qua has formulas indagavimus, etiam multo latius patet atque ad alias curvas loco ellipsis assumtas extendi poterit. Si enim coordinatae pro curva data fuerint

$$X=2f\cos.\alpha\theta+2f'\cos.\beta\theta+\text{etc.},$$
$$Y=2g\sin.\alpha\theta+2g'\sin.\beta\theta+\text{etc.},$$

pro reliquis curvis cum proposita communem rectificationem habentibus ponendo iterum

$$f-g=a, \quad f+g=b \quad \text{et} \quad f'-g'=a', \quad f'+g'=b' \quad \text{etc.}$$

fiet

$$\begin{aligned}x=&\frac{\alpha a}{n+\alpha}\cos.(n+\alpha)\theta-\frac{\alpha b}{n-\alpha}\cos.(n-\alpha)\theta\\&+\frac{\beta a'}{n+\beta}\cos.(n+\beta)\theta-\frac{\beta b'}{n-\beta}\cos.(n-\beta)\theta+\text{etc.},\\y=&\frac{\alpha a}{n+\alpha}\sin.(n+\alpha)\theta-\frac{\alpha b}{n-\alpha}\sin.(n-\alpha)\theta\\&+\frac{\beta a'}{n+\beta}\sin.(n+\beta)\theta-\frac{\beta b'}{n-\beta}\sin.(n-\beta)\theta+\text{etc.}\end{aligned}$$

Ubi iterum ob n numerum indefinitum innumerabiles curvae prodeunt.

DE INFINITIS CURVIS ALGEBRAICIS QUARUM LONGITUDO ARCUI PARABOLICO AEQUATUR

Convent. exhib. die 20 Augusti 1781

Commentatio 781 indicis ENESTROEMIANI

Mémoires de l'académie des sciences de St. Pétersbourg 11, 1830, p. 100—101

PROBLEMA

Proposita parabola AYC (Fig. 1, p. 247) *ad axem* AB *relata, cuius para-* *sit* $AB = BC$, *invenire innumeras curvas algebraicas* AZ, *quarum arcus* AZ *les sint arcui parabolico* AY.

CONSTRUCTIO

Ad axem AB retro productam in F usque eadem describatur parabola AG. In hoc axe capiatur pro lubitu punctum F, ita tamen ut ducta applicata FG haec recta FG ad parametrum AB rationem teneat rationalem, quae sit $\frac{AB}{FG} = n$. Tum enim ex quolibet tali puncto F construi poterit una curva AZ quaestioni satisfaciens.

Pro parabolae enim puncto quocunque Y abscissa AX et applicata XY determinato rectae FG normaliter iungatur $GV = XY$, ut obtineatur angulus $GFV = \theta$; quo invento capiatur angulus $AFZ = n\theta$ sumaturque $FZ = FX$ eritque Z punctum in curva quaesita, cuius arcus AZ aequalis erit arcui AY. Hoc igitur modo, cum punctum F infinitis modis assumi possit, construentur innumerae curvae AZ eiusdem indolis eademque proprietate gaudentes.

DEMONSTRATIO

Posito $AB = BC = 2a$ sit $AX = x$ et $XY = y$ ideoque $yy = 2ax$, unde fit $\partial x = \frac{y\partial y}{a}$ et elementum parabolae

$$\partial s = \partial y\sqrt{1 + \frac{yy}{aa}}.$$

Iam ponatur $AF = f$ et $FG = g$; erit quoque $gg = 2af$. Iam vocetur $FZ = FX = f + x = z$ atque angulus $AFZ = \varphi$ eritque elementum curvae quaesitae $= \sqrt{\partial z^2 + zz\partial\varphi^2}$. Fieri ergo debet

$$\partial z^2 + zz\partial\varphi^2 = \partial y^2 + \frac{yy\partial y^2}{aa}.$$

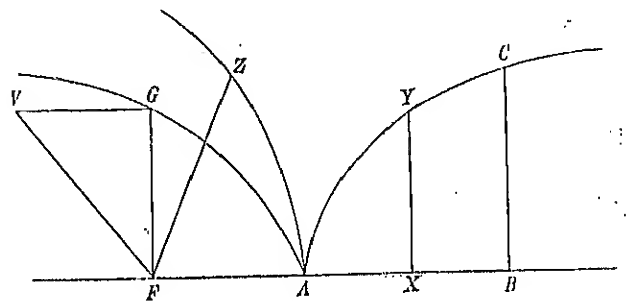

Fig. 1.

Cum igitur sit $\partial z = \partial x = \frac{y\partial y}{a}$, fiet $zz\partial\varphi^2 = \partial y^2$ ideoque $\partial\varphi = \frac{\partial y}{z}$. Est vero $f = \frac{gg}{2a}$ et $x = \frac{yy}{2a}$, ergo $\partial\varphi = \frac{2a\partial y}{gg + yy}$, consequenter $\varphi = \frac{2a}{g}$ Arc. tang. $\frac{y}{g}$[1]. At vero est Arc. tang. $\frac{y}{g} = \theta$ et $\frac{2a}{g} = n$ ideoque $\varphi = n\theta$. Sumto ergo angulo $AFZ = n\theta$ et recta $FZ = FX$ punctum Z in tali erit curva, cuius elementum elemento parabolae aequatur.

1) In editione principe Commentationum 781—783 loco tang. semper scriptum est tag.
A. K.

DE BINIS CURVIS ALGEBRAICIS EADEM RECTIFICATIONE GAUDENTIBUS

Convent. exhib. die 20 Augusti 1781

Commentatio 782 indicis ENESTROEMIANI

Mémoires de l'académie des sciences de St. Pétersbourg 11, 1830, p. 102—113

1. *Sint x et y coordinatae orthogonales unius, at X et Y alterius curvae et quaestio eo redit, ut fiat*

$$\partial x^2 + \partial y^2 = \partial X^2 + \partial Y^2,$$

ita tamen, ut omnes expressiones prodeant algebraicae.

Huius igitur problematis duplicem hic sum traditurus solutionem; quae cum plurimum a se invicem discrepare videantur, earum quoque consensum ostendere conveniet.

SOLUTIO PRIOR

2. Cum igitur reddi oporteat $\partial X^2 + \partial Y^2 = \partial x^2 + \partial y^2$, hoc praestabitur, si statuamus

$$\partial X = \partial x \cos.\varphi + \partial y \sin.\varphi,$$
$$\partial Y = \partial x \sin.\varphi - \partial y \cos.\varphi,$$

ubi ergo angulum φ ita comparatum esse necesse est, ut hae duae formulae integrationem admittant. Ad hoc efficiendum utar methodo olim[1]) a me tradita, ubi prima quasi elementa Analyseos infinitorum indeterminatae exposui. Tum igitur prodibit

1) L. EULERI Commentatio 245 (indicis ENESTROEMIANI): *De methodo DIOPHANTEAE analoga in analysi infinitorum*, Novi comment. acad. sc. Petrop. **5** (1754/5), 1760, p. 84; *LEONHARDI EULERI Opera omnia*, series I, vol. 22. A. K.

$$X = x\cos.\varphi + y\sin.\varphi + \int \partial\varphi (x\sin.\varphi - y\cos.\varphi),$$
$$Y = x\sin.\varphi - y\cos.\varphi - \int \partial\varphi (x\cos.\varphi + y\sin.\varphi),$$

ubi ergo has duas formulas integrales integrabiles reddi oportet, id quod nulla difficultate laborat.

3. Statuamus enim

$$\int \partial\varphi (x\sin.\varphi - y\cos.\varphi) = P, \quad \int \partial\varphi (x\cos.\varphi + y\sin.\varphi) = Q$$

eritque

$$x\sin.\varphi - y\cos.\varphi = \frac{\partial P}{\partial\varphi}, \quad x\cos.\varphi + y\sin.\varphi = \frac{\partial Q}{\partial\varphi},$$

ubi ergo pro P et Q functiones quascunque algebraicas ipsarum $\sin.\varphi$ et $\cos.\varphi$ accipere licet. Tum vero ex his duabus aequationibus ipsae coordinatae x et y sequenti modo determinantur

$$x = \frac{\partial P\sin.\varphi + \partial Q\cos.\varphi}{\partial\varphi},$$
$$y = \frac{\partial Q\sin.\varphi - \partial P\cos.\varphi}{\partial\varphi}.$$

Ex quibus iam coordinatae alterius curvae sponte determinantur

$$X = \frac{\partial Q}{\partial\varphi} + P, \quad Y = \frac{\partial P}{\partial\varphi} - Q.$$

Hinc ergo nullo plane labore innumerabilia binarum curvarum algebraicarum paria exhiberi poterunt, quae eadem rectificatione erunt praeditae.

4. Quo hoc clarius appareat, sumamus differentialia capiendo $\partial\varphi$ constante ac reperietur

$$\partial x = \frac{\partial\partial P\sin.\varphi + \partial\partial Q\cos.\varphi}{\partial\varphi} + \partial P\cos.\varphi - \partial Q\sin.\varphi,$$
$$\partial y = \frac{\partial\partial Q\sin.\varphi - \partial\partial P\cos.\varphi}{\partial\varphi} + \partial Q\cos.\varphi + \partial P\sin.\varphi,$$

unde colligitur

$$\partial x^2 + \partial y^2 = \frac{\partial\partial P^2 + \partial\partial Q^2}{\partial\varphi^2} + \frac{2(\partial P\partial\partial Q - \partial Q\partial\partial P)}{\partial\varphi} + \partial P^2 + \partial Q^2.$$

Simili modo pro altera curva habebimus

$$\partial X = \frac{\partial\partial Q}{\partial\varphi} + \partial P \quad \text{et} \quad \partial Y = \frac{\partial\partial P}{\partial\varphi} - \partial Q,$$

ex quibus pro arcus elemento erit

$$\partial X^2 + \partial Y^2 = \frac{\partial\partial Q^2 + \partial\partial P^2}{\partial\varphi^2} + \frac{2(\partial P\partial\partial Q - \partial Q\partial\partial P)}{\partial\varphi} + \partial P^2 + \partial Q^2$$

ideoque

$$\partial X^2 + \partial Y^2 = \partial x^2 + \partial y^2,$$

uti requiritur.

SOLUTIO POSTERIOR

5. Cum effici debeat $\partial x^2 + \partial y^2 = \partial X^2 + \partial Y^2$, erit $\partial x^2 - \partial X^2 = \partial Y^2 - \partial y^2$, ad quam aequationem resolvendam statuamus

$$x + X = M, \quad x - X = m, \quad Y + y = N, \quad Y - y = n,$$

quo facto fieri debet $\partial M \partial m = \partial N \partial n$, consequenter $\frac{\partial M}{\partial n} = \frac{\partial N}{\partial m}$; quarum duarum fractionum utraque ponatur $= t$, ut habeamus primo $\partial M = t\partial n$ ideoque

$$M = tn - \int n\partial t.$$

Simili modo pro altera erit $\partial N = t\partial m$, ergo

$$N = tm - \int m\partial t.$$

6. Hoc igitur modo novam variabilem t in calculum introduximus, ex qua ipsas coordinatas facile definire licebit. Ponamus enim $\int n\partial t = U$, ut fiat

$$n = \frac{\partial U}{\partial t} \quad \text{hincque} \quad M = \frac{t\partial U}{\partial t} - U.$$

Simili modo ponendo $\int m\partial t = V$ habebimus

$$m = \frac{\partial V}{\partial t}, \quad \text{hinc} \quad N = \frac{t\partial V}{\partial t} - V,$$

ubi U et V denotent functiones quascunque ipsius t.

7. Ex his iam valoribus ipsae coordinatae utriusque curvae sponte se produnt. Cum enim sit

$$x = \frac{M+m}{2}, \qquad X = \frac{M-m}{2}, \qquad Y = \frac{N+n}{2}, \qquad y = \frac{N-n}{2},$$

nihil impedit, quominus has formulas duplicemus, hincque coordinatae utriusque curvae sequenti modo exprimentur

$$x = \frac{t\partial U - U\partial t + \partial V}{\partial t}, \qquad X = \frac{t\partial U - U\partial t - \partial V}{\partial t},$$

$$y = \frac{t\partial V - V\partial t - \partial U}{\partial t}, \qquad Y = \frac{t\partial V - V\partial t + \partial U}{\partial t}.$$

8. Videamus nunc etiam, quomodo hae formulae quaestioni propositae satisfaciant. Ac sumto elemento ∂t constante elementa pro priore curva erunt

$$\partial x = \frac{t\partial\partial U + \partial\partial V}{\partial t}, \quad \partial y = \frac{t\partial\partial V - \partial\partial U}{\partial t},$$

unde fit

$$\partial x^2 + \partial y^2 = \frac{(1+tt)(\partial\partial U^2 + \partial\partial V^2)}{\partial t^2}.$$

Pro altera curva habebimus

$$\partial X = \frac{t\partial\partial U - \partial\partial V}{\partial t}, \quad \partial Y = \frac{t\partial\partial V + \partial\partial U}{\partial t}$$

hincque

$$\partial X^2 + \partial Y^2 = \frac{(1+tt)(\partial\partial U^2 + \partial\partial V^2)}{\partial t^2}.$$

9. Quamquam hae duae solutiones toto coelo a se invicem discrepare videntur, tamen nullum dubium, quin inter se pulcherrime consentiant, cum utraque omnes plane casus satisfacientes complecti debeat. Interim tamen,

si solutiones simpliciores desideremus, prior ad hunc scopum magis apta deprehenditur, quippe quae ita restricta, ut ponatur $Q=0$, adhuc plurimas solutiones memorabiles suppeditat. Posito autem $Q=0$ coordinatae binarum curvarum per formulas istas simplicissimas exprimentur

$$x = \frac{\partial P \sin.\varphi}{\partial\varphi}, \qquad X = P,$$
$$y = -\frac{\partial P \cos.\varphi}{\partial\varphi}, \qquad Y = \frac{\partial P}{\partial\varphi}.$$

Ubi cum sit $P=X$, adeo immediate ex posteriore curva ad priorem procedere licebit, ita ut altera curvarum quaesitarum nunc quasi cognita spectari possit, id quod in formulis generalibus nullo modo fieri potest. Hanc igitur solutionem, etsi maxime particularem, fusius prosequi conveniet, ubi quidem litteras maiusculas et minusculas inter se permutemus.

SOLUTIO PARTICULARIS HAS COORDINATAS COMPLECTENS

$$x = P \qquad X = \frac{\partial P \sin.\varphi}{\partial\varphi}$$
$$y = \frac{\partial P}{\partial\varphi} \qquad Y = \frac{\partial P \cos.\varphi}{\partial\varphi}$$

10. Cum hic pro priore curva sit $P=x$, erit $y=\frac{\partial x}{\partial\varphi}$, unde fit $\partial\varphi=\frac{\partial x}{y}$; cum igitur $\partial\omega$ sit elementum arcus circularis, quoties aequatio inter x et y ita fuerit comparata, ut formula integralis $\int\frac{\partial x}{y}$ arcum circularem exprimat, toties alia curva exhiberi poterit eandem rectificationem involvens, quippe pro qua habebitur

$$1.\quad \frac{X}{Y} = \text{tang.}\,\varphi\,^{1)};$$

deinde quoque habebitur

$$2.\quad \sqrt{X^2+Y^2} = \frac{\partial x}{\partial\varphi} = y,$$

ita ut chorda curvae quaesitae semper aequalis sit applicatae alterius curvae. Tales igitur casus accuratius evolvere operae erit pretium.

1) Vide notam p. 247. A. K.

EVOLUTIO CASUS
QUO PRO CURVA DATA EST $y = \frac{aa + xx}{b}$

11. Hic statim patet istam aequationem pertinere ad parabolam, cuius parameter $= b$, eamque adeo permanere eandem, utcunque quantitas a immutetur, cum tantum initium applicatarum mutetur, quamobrem, si curva quaesita ab a pendebit, hinc infinitae adeo curvae diversae reperientur, quae cum parabola communi gaudeant rectificatione.

12. Hinc igitur fiet $\partial\varphi = \frac{b\partial x}{aa + xx}$, ubi ponamus $b = na$, ut integrando prodeat $\varphi = n \,\mathrm{A\,tang.}\, \frac{x}{a}$. Quia igitur volumus, ut parameter b invariatus maneat, erit $a = \frac{b}{n}$ sive $n = \frac{b}{a}$, ita ut numerus n rationem inter parametrum b et quantitatem arbitrariam a involvat. Hinc igitur fiet $x = a \,\mathrm{tang.}\, \frac{\varphi}{n}$. Unde patet, ut formulae nostrae prodeant algebraicae, numerum n absolute rationalem esse debere; alioquin enim ad genus quantitatum, quae interscendentes appellari solent, devolveremur.

13. Cum igitur hinc sit

$$\partial x = \frac{a\partial\varphi}{n \cos. \frac{\varphi}{n}^2},$$

erit pro curva quaesita

$$\sqrt{X^2 + Y^2} = y \quad \text{et} \quad \frac{X}{Y} = \mathrm{tang.}\, \varphi.$$

Quia igitur angulus φ ex ipsa aequatione pro curva data innotescit, haec curva facile geometrice construi poterit atque constructio eadem plane prodit, quam non ita pridem¹) pro infinitis curvis algebraicis, quae cum parabola communem rectificationem habeant, dedi.

EVOLUTIO CASUS
QUO PRO CURVA DATA EST $ny = \sqrt{aa - xx}$

14. Hic igitur erit

$$\partial\varphi = \frac{\partial x}{y} = \frac{n\partial x}{\sqrt{aa - xx}}$$

1) L. EULERI Commentatio 781 (indicis ENESTROEMIANI); vide p. 246. A. K.

ideoque $\varphi = n\,\text{A}\sin.\frac{x}{a}$, unde fit

$$x = a\sin.\frac{\varphi}{n} \quad \text{et} \quad y = \frac{a}{n}\cos.\frac{\varphi}{n}.$$

Evidens autem est hanc curvam datam esse ellipsin, cuius alter semiaxis $= a$, alter vero $\frac{a}{n}$. Pro curva quaesita igitur habebimus eius chordam

$$\sqrt{X^2 + Y^2} = \frac{a}{n}\cos.\frac{\varphi}{n} \quad \text{et} \quad \frac{X}{Y} = \text{tang.}\,\varphi,$$

unde iterum constructio facillima deducitur, si modo n fuerit numerus rationalis. Cognita enim chorda et angulo, quo ea ad axem fixum inclinatur, constructio facillime expedietur.

15. Hic ante omnia observasse iuvabit, si pro data curva circulum accipiamus, ut sit $n = 1$, fore $y = \sqrt{aa - xx}$. Ponamus brevitatis gratia

$$\sqrt{X^2 + Y^2} = Z,$$

et cum sit $y = Z = \sqrt{aa - xx}$, erit $x = \sqrt{aa - ZZ}$ hincque

$$\text{tang.}\,\varphi = \frac{X}{Y} = \frac{x}{y} = \frac{\sqrt{aa - ZZ}}{Z}.$$

Hinc fiet $\frac{X^2}{Y^2} = \frac{aa - ZZ}{ZZ}$ sive $ZZ(XX + YY) = aaYY$ seu $Z^4 = aaYY$ atque $ZZ = aY$, quae est aequatio pro circulo, ita ut etiam nunc nulla curva exhiberi posse videatur, quae cum circulo communi rectificatione gaudeat praeter ipsum circulum.

16. Consideremus etiam casum, quo $n = 2$, quo fit

$$x = a\sin.\frac{\varphi}{2} \quad \text{et} \quad y = \frac{a}{2}\cos.\frac{\varphi}{2}.$$

Hinc igitur erit $Z = \frac{a}{2}\cos.\frac{\varphi}{2}$. Cum igitur sit $\text{tang.}\,\varphi = \frac{X}{Y}$, erit $\cos.\varphi = \frac{Y}{Z}$. Cum autem $\cos.\frac{1}{2}\varphi = \sqrt{\frac{1 + \cos.\varphi}{2}}$, pro curva quaesita oritur haec aequatio

$$Z = \frac{a}{2}\sqrt{\frac{Z+Y}{2Z}} \quad \text{ideoque} \quad 8Z^3 = aa(Z+Y),$$

quae expressio ob $Z = \sqrt{XX+YY}$ ad rationalitatem perducta ad gradum sextum ascendit.

EVOLUTIO CASUS QUO PRO CURVA DATA EST $ny = b + \sqrt{aa - xx}$

17. Evidens est hanc aequationem semper esse pro ellipsi, quicunque valor litterae b tribuatur, atque adeo casu $n = 1$ hanc curvam fore circulum. Tum autem habebimus

$$\partial\varphi = \frac{n\partial x}{b + \sqrt{aa - xx}},$$

quae expressio posito $x = \frac{2au}{1+uu}$, unde fit

$$\partial x = \frac{2a\partial u(1-uu)}{(1+uu)^2} \quad \text{et} \quad \sqrt{aa-xx} = \frac{a(1-uu)}{1+uu},$$

induit hanc formam

$$\partial\varphi = \frac{2na\partial u(1-uu)}{(1+uu)(b+a+uu(b-a))},$$

quam in duas huiusmodi partes discerpere licet

$$\frac{\alpha\partial u}{1+uu} + \frac{\beta\partial u}{b+a+(b-a)uu},$$

quarum integratio utraque ad arcum circuli deducitur, si modo fuerit $b > a$.

18. Resolutione autem facta reperitur $\alpha = 2n$ et $\beta = -2nb$, ita ut habeamus

$$\partial\varphi = \frac{2n\partial u}{1+uu} - \frac{2nb\partial u}{b+a+(b-a)uu}.$$

Cum iam in genere sit

$$\int \frac{\partial u}{f+guu} = \frac{1}{\sqrt{fg}} \text{A tang.} \frac{u\sqrt{g}}{\sqrt{f}},$$

erit

$$\varphi = 2n \text{ A tang. } u - \frac{2nb}{\sqrt{bb-aa}} \text{ A tang. } u \sqrt{\frac{b-a}{b+a}}.$$

Haec igitur aequatio ut primo fiat realis, necesse est, ut sit $b > a$; deinde ut etiam algebraica fiat, necesse est, ut tam $2n$ quam $\frac{2nb}{\sqrt{bb-aa}}$ sint numeri rationales. Hunc in finem eiusmodi rationem inter b et a statui oportet, ut fiat $\frac{b}{\sqrt{bb-aa}} = \lambda$ numerus rationalis, unde fit $\frac{b}{a} = \frac{\lambda}{\sqrt{\lambda\lambda-1}}$, sicque erit

$$\frac{b-a}{b+a} = \frac{\lambda - \sqrt{\lambda\lambda-1}}{\lambda+\sqrt{\lambda\lambda-1}} = \frac{1}{(\lambda+\sqrt{\lambda\lambda-1})^2} \quad \text{ideoque} \quad \sqrt{\frac{b-a}{b+a}} = \frac{1}{\lambda+\sqrt{\lambda\lambda-1}},$$

quo valore substituto fiet

$$\varphi = 2n \text{ A tang. } u - 2n\lambda \text{ A tang. } \frac{u}{\lambda+\sqrt{\lambda\lambda-1}}.$$

19. Componitur ergo angulus φ ex duobus angulis, quos vocemus ζ et η, quorumque ergo tangentes per u ita exprimuntur, ut sit

$$\text{tang. } \zeta = u \quad \text{et} \quad \text{tang. } \eta = \frac{u}{\lambda+\sqrt{\lambda\lambda-1}};$$

tum vero erit

$$\varphi = 2n\zeta - 2n\lambda\eta \quad \text{sive} \quad \frac{\varphi}{2n} = \zeta - \lambda\eta.$$

Nunc evidens est, si modo λ fuerit numerus rationalis, etiam anguli $\lambda\eta$ tangentem algebraice per u exprimi; ergo etiam tangens differentiae horum angulorum, hoc est anguli $\frac{\varphi}{2n}$, aequabitur functioni algebraicae ipsius u ideoque etiam tangens ipsius anguli φ, si modo n fuerit numerus rationalis, unde patet hanc solutionem ad alias ellipses adaptari non posse.

20. Cum igitur ellipsis, quam consideremus, eadem maneat, quicunque valor ipsi b tribuatur, ad eius indolem cognoscendam sumamus $b = 0$, ut sit $y = \frac{\sqrt{aa-xx}}{n}$, unde patet eius semiaxem transversum fore $= a$, ubi scilicet $y = 0$, coniugatum vero $= \frac{a}{n}$. Quare noster calculus ad alias ellipses accommodari nequit, nisi quarum axes inter se teneant rationem rationalem. Praeterea vero pro b alios valores assumere non licet, nisi quibus fit $\frac{b}{\sqrt{bb-aa}}$ numerus rationalis. Unde patet nihilominus semper innumeras curvas algebraicas inveniri posse, quae cum tali ellipsi communem rectificationem contineant.

21. Cum igitur pro curva quaesita sit $\frac{X}{Y} = \operatorname{tang.}\varphi$, etiam haec fractio $\frac{X}{Y}$ per functionem algebraicam ipsius u exprimetur. Deinde, quia invenimus

$$\sqrt{X^2 + Y^2} = y = \frac{b + \sqrt{aa - xx}}{n},$$

etiam haec chorda per functionem algebraicam ipsius u exprimetur, cum sit

$$x = \frac{2au}{1+uu} \quad \text{et} \quad \sqrt{aa - xx} = \frac{a(1-uu)}{1+uu},$$

unde fit

$$\sqrt{X^2 + Y^2} = \frac{b + a + (b-a)uu}{n(1+uu)}.$$

Quamobrem cum ambae hae formulae $\frac{X}{Y}$ et $\sqrt{X^2+Y^2}$ per functiones algebraicas eiusdem quantitatis u determinentur, eliminando hanc quantitatem u, id quod facile fit ex valore ipsius $\sqrt{X^2+Y^2}$, quippe quo posito $= Z$ colligitur $uu = \frac{b+a-nZ}{nZ-(b-a)}$. Hic igitur valor in formula pro $\operatorname{tang.}\varphi$ inventa, cui $\frac{X}{Y}$ aequatur, substitutus praebebit aequationem algebraicam inter binas coordinatas curvae quaesitae X et Y ob $Z = \sqrt{X^2+Y^2}$, quae autem plerumque ad plurimas dimensiones exsurget.

22. Hic probe notandum est, quoniam (vid. Nov. Act. t. 5[1])) infinitas curvas algebraicas determinavi, quae cum data ellipsi quacunque communi gaudeant rectificatione solo circulo excepto, eas curvas ab iis, quas nunc invenimus, prorsus esse diversas; neque etiam patet, quomodo illae ex solutione particulari, qua hic usi sumus, deduci queant. Facile autem derivari possunt ex formulis generalibus primae solutionis, id quod hic ostendisse operae pretium videtur.

23. Quia ibi pro altera curva dedimus hos valores

$$x = \frac{\partial P \sin.\varphi + \partial Q \cos.\varphi}{\partial\varphi} \quad \text{et} \quad y = \frac{\partial Q \sin.\varphi - \partial P \cos.\varphi}{\partial\varphi},$$

sumamus

$$\frac{\partial P}{\partial\varphi} = -a\cos.(n+1)\varphi + b\cos.(n-1)\varphi$$

et

$$\frac{\partial Q}{\partial\varphi} = a\sin.(n+1)\varphi + b\sin.(n-1)\varphi$$

1) L. EULERI Commentatio 639 (indicis ENESTROEMIANI); vide p. 163. A. K.

eritque

$$x = (a + b)\sin. n\varphi \quad \text{et} \quad y = (a - b)\cos. n\varphi,$$

unde manifesto fit

$$\frac{xx}{(a+b)^2} + \frac{yy}{(a-b)^2} = 1,$$

quae aequatio est pro ellipsi, cuius semiaxes sunt $a + b$ et $a - b$.

24. Ex his autem valoribus differentialibus colligitur integrando

$$P = -\frac{a\sin.(n+1)\varphi}{n+1} + \frac{b\sin.(n-1)\varphi}{n-1},$$
$$Q = -\frac{a\cos.(n+1)\varphi}{n+1} - \frac{b\cos.(n-1)\varphi}{n-1}.$$

Quare, cum pro altera curva invenerimus

$$X = \frac{\partial Q}{\partial\varphi} + P \quad \text{et} \quad Y = \frac{\partial P}{\partial\varphi} - Q,$$

isti valores ita se habebunt

$$X = \frac{na}{n+1}\sin.(n+1)\varphi + \frac{nb}{n-1}\sin.(n-1)\varphi,$$
$$Y = -\frac{na}{n+1}\cos.(n+1)\varphi + \frac{nb}{n-1}\cos.(n-1)\varphi.$$

Unde patet, quoniam numerus n penitus arbitrio nostro relinquitur, ex his formulis infinitas prodire curvas algebraicas nulla alia conditione restrictas, nisi ut n sit numerus rationalis exceptis tantum duobus casibus $n = 1$ et $n = -1$; simul vero intelligitur, utcunque ratio inter axes fuerit irrationalis, curvas quaesitas non turbari.

PROBLEMA

25. *Consensum inter ambas solutiones generales monstrare et substitutiones indagare, quibus altera in alteram converti queat.*

SOLUTIO

Quoniam in formulis supra datis tam coordinatas quam functiones inter se permutare licet, ad calculi commoditatem priores coordinatas x et y

sequenti modo repraesentemus

pro priore solutione	pro posteriore solutione
$x = \dfrac{\partial P \cos.\varphi - \partial Q \sin.\varphi}{\partial \varphi}$	$x = \dfrac{\partial U}{\partial t} - \dfrac{t\partial V}{\partial t} + V$
$y = \dfrac{\partial P \sin.\varphi + \partial Q \cos.\varphi}{\partial \varphi}$	$y = \dfrac{\partial V}{\partial t} + \dfrac{t\partial U}{\partial t} - U.$

Hic igitur ostendendum, qualem relationem primo inter φ et t, deinde vero inter functiones P, Q et V, U statui oporteat, ut isti duplices valores ipsarum x et y ad identitatem revocentur.

26. Hunc in finem ante omnia necesse est multitudinem quantitatum, quae hic occurrunt, imminuere, id quod pulcherrime succedit, si pro priore solutione statuamus

$$P + Q\sqrt{-1} = \Theta;$$

tum enim fiet

$$x + y\sqrt{-1} = \frac{\cos.\varphi + \sqrt{-1}\sin.\varphi}{\partial \varphi}\,\partial\Theta.$$

Pro altera vero solutione ponamus

$$U + V\sqrt{-1} = \Pi$$

ac reperietur

$$x + y\sqrt{-1} = \frac{\partial \Pi}{\partial t}(1 + t\sqrt{-1}) - \Pi\sqrt{-1}.$$

Haec autem expressio ad hanc formam redigitur

$$x + y\sqrt{-1} = \frac{(1 + t\sqrt{-1})^2}{\partial t}\,\partial.\frac{\Pi}{1 + t\sqrt{-1}}.$$

Totum negotium ergo huc redit, ut hae duae formulae pro $x + y\sqrt{-1}$ inventae consentientes reddantur.

27. Quo factores priores ad maiorem uniformitatem revocemus, ponamus $t = \text{tang.}\,\omega$ eritque

$$1 + t\sqrt{-1} = \frac{\cos.\omega + \sqrt{-1}\sin.\omega}{\cos.\omega} \quad \text{et} \quad \partial t = \frac{\partial\omega}{\cos.\omega^2},$$

unde fit

$$\frac{(1+t\sqrt{-1})^2}{\partial t} = \frac{\cos. 2\omega + \sqrt{-1}\sin. 2\omega}{\partial\omega}.$$

Quamobrem nunc ista aequalitas erit docenda

$$\frac{\cos.\varphi + \sqrt{-1}\sin.\varphi}{\partial\varphi}\partial\Theta = \frac{\cos. 2\omega + \sqrt{-1}\sin. 2\omega}{\partial\omega}\partial.\frac{\Pi\cos.\omega}{\cos.\omega + \sqrt{-1}\sin.\omega}$$

et nunc evidens est statui debere $\varphi = 2\omega$; tum enim dividendo utrinque per $\frac{\cos. 2\omega + \sqrt{-1}\sin. 2\omega}{\partial\omega}$ orietur ista aequalitas satis simplex

$$\frac{1}{2}\partial\Theta = \partial.\frac{\Pi\cos.\omega}{\cos.\omega + \sqrt{-1}\sin.\omega}.$$

Integralibus igitur sumendis debet esse $\Theta = \frac{2\Pi\cos.\omega}{\cos.\omega + \sqrt{-1}\sin.\omega}$ sive

$$\Theta(\cos.\omega + \sqrt{-1}\sin.\omega) = 2\Pi\cos.\omega.$$

28. Restituamus nunc loco Θ et Π valores assumtos orieturque haec aequatio

$$(P + Q\sqrt{-1})(\cos.\omega + \sqrt{-1}\sin.\omega) = 2\cos.\omega(U + V\sqrt{-1}),$$

unde partes reales et imaginarias seorsim inter se aequari oportet, hincque ergo duae sequentes determinationes deducuntur

$$2U\cos.\omega = P\cos.\omega - Q\sin.\omega,$$
$$2V\cos.\omega = P\sin.\omega + Q\cos.\omega,$$

ubi meminisse oportet esse $t = \text{tang.}\,\omega$ et $\varphi = 2\omega$, sicque si in solutione posteriore loco U et V isti valores substituantur

$$U = \frac{P\cos.\omega - Q\sin.\omega}{2\cos.\omega} \quad \text{et} \quad V = \frac{P\sin.\omega + Q\cos.\omega}{2\cos.\omega},$$

ea in priorem convertetur.

29. Vicissim igitur functiones P et Q per U et V ita definientur

$$P = 2U\cos.\omega^2 + 2V\sin.\omega\cos.\omega$$

sive
$$P = U(1 + \cos. 2\omega) + V \sin. 2\omega$$
et
$$Q = V(1 + \cos. 2\omega) - U \sin. 2\omega.$$

Hoc igitur modo patet non solum binas expressiones perfecte inter se consentire, sed etiam substitutiones habentur, quibus altera in alteram converti potest.

30. Ostendamus igitur clarius, quomodo posteriores formulae ad priores reduci debeant. Ac primo quidem cum sit
$$t = \text{tang. } \omega = \text{tang. } \frac{1}{2}\varphi,$$
erit
$$t = \frac{\sin. \varphi}{1 + \cos. \varphi} \quad \text{et} \quad \partial t = \frac{\partial \varphi}{1 + \cos. \varphi};$$
tum vero erit etiam
$$U = \frac{P}{2} - \frac{Q \sin. \varphi}{2(1 + \cos. \varphi)} \quad \text{et} \quad V = \frac{Q}{2} + \frac{P \sin. \varphi}{2(1 + \cos. \varphi)}.$$
Simili modo priores ex posterioribus nascentur; namque ob tang. $\frac{1}{2}\varphi = t$ erit
$$\sin. \varphi = \frac{2t}{1 + tt} \quad \text{et} \quad \cos. \varphi = \frac{1 - tt}{1 + tt},$$
tum vero
$$\partial \varphi = \frac{2 \partial t}{1 + tt};$$
functiones vero P et Q ita definientur, ut sit
$$P = \frac{2U + 2Vt}{1 + tt} \quad \text{et} \quad Q = \frac{2V - 2Ut}{1 + tt}.$$

31. Sufficiet autem consensum inter formulas binas pro coordinatis x et y ostendisse, quandoquidem nullum dubium superesse potest, quin per has substitutiones etiam formulae pro coordinatis X et Y alterae in alteras convertantur, atque hoc modo quaestioni principali, quam hic tractare suscepimus, perfecte est satisfactum, dum nostrae formulae omnia binarum curvarum algebraicarum paria largiuntur, quae eadem rectificatione sint praeditae.

DE CURVIS ALGEBRAICIS QUARUM OMNES ARCUS PER ARCUS CIRCULARES METIRI LICEAT

Convent. exhib. die 20 Augusti 1781

Commentatio 783 indicis ENESTROEMIANI

Mémoires de l'académie des sciences de St. Pétersbourg 11, 1830, p. 114—124

1. Non dubitavi ante aliquot annos[1]) istam propositionem tanquam insigne theorema in medium proferre: *quod praeter circulum nulla detur curva algebraica, cuius arcubus omnibus aequales arcus circulares assignari queant.* Plures etiam adduxi rationes satis probabiles, quae me in hac opinione confirmabant, quanquam probe perspexi eas a perfecta demonstratione adhuc plurimum distare. Praecipua autem ratio mihi erat, quod, postquam in hoc argumento plurimum elaborassem, nullam tamen huiusmodi curvam elicere potuerim.

2. Quamobrem, cum nuper in simili argumento occupatus in genere binas curvas algebraicas investigassem, quae communi rectificatione gauderent, indeque infinitas curvas algebraicas investigassem, quarum longitudo per arcus parabolicos metiri liceret, tum vero etiam infinitas curvas algebraicas cum ellipsi eadem rectificatione gaudentes, maxime obstupui, quod, etiamsi ellipsin in circulum converterem, nihilominus curvae inventae a circulo essent diversae. Sententiam igitur meam hic solenniter retractans methodum facilem exponam, cuius ope innumerabiles curvae algebraicae inveniri possunt, quarum omnes arcus circularibus sunt aequales.

3. Proposito igitur circulo centro c (Fig. 1 et 2, p. 263) radio ca [$= 1$] descripto concipiamus curvam AZ ita comparatam, ut eius arcus indefinitus AZ

1) Vide p. 83. A. K.

semper aequalis sit arcui indefinito illius circuli az, quo vocato $az = \omega$ sit quoque arcus $AZ = \omega$. Hanc iam curvam ad centrum quoddam fixum C refero eiusque naturam per aequationem inter distantiam $CZ = z$ et angulum $ACZ = \varphi$ investigabo, ut quaesito satisfiat. Cum igitur hinc sit arcus $AZ = \int \sqrt{\partial z^2 + zz\partial\varphi^2}$, fieri debet $\partial\omega^2 = \partial z^2 + zz\partial\varphi^2$, unde deducitur

$$\partial\varphi = \frac{\sqrt{\partial\omega^2 - \partial z^2}}{z},$$

ubi ergo totum negotium huc redit, ut eiusmodi relatio inter z et ω exquiratur, quae integrale huius formulae $\varphi = \int \frac{\sqrt{\partial\omega^2 - \partial z^2}}{z}$ per arcum circularem simpliciter exprimat.

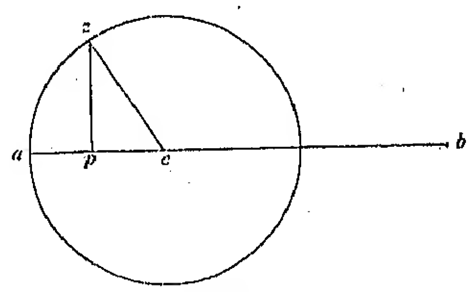

Fig. 1.

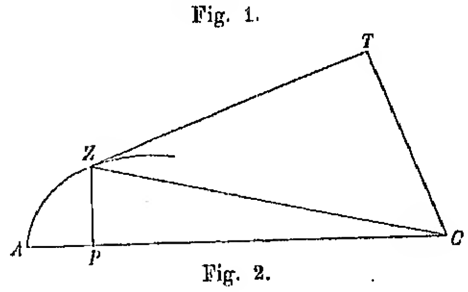

Fig. 2.

4. Observavi autem hoc satis commode praestari posse, si statuamus distantiam $CZ = b + \cos.\omega$; quem in finem sumo intervallum $cb = b$ ac demisso ex z perpendiculo zp fiet $cp = \cos.\omega$ sicque distantia CZ semper aequalis capi debet intervallo bp. Unde patet pro initio A nostrae curvae fore distantiam $CA = ba = b + 1$. Cum igitur hinc fiat $\partial z = -\partial\omega \sin.\omega$, formula differentialis pro $\partial\varphi$ data posito $z = b + \cos.\omega$ induet hanc formam satis concinnam

$$\partial\varphi = \frac{\partial\omega \cos.\omega}{b + \cos.\omega},$$

cuius ergo integrale arcui circulari aequale esse debet.

5. Ista autem formula sponte in has partes discerpitur $\partial\varphi = \partial\omega - \frac{b\partial\omega}{b + \cos\omega}$, quarum prima per se est elementum circuli. Pro altera parte ponamus

$$\operatorname{tang.} \frac{1}{2}\omega = t^{1)}$$

1) Vide notam p. 247. A. K.

fietque

$$\partial\omega = \frac{2\partial t}{1+tt};$$

tum vero fit

$$\sin.\frac{1}{2}\omega = \frac{t}{\sqrt{1+tt}} \quad \text{et} \quad \cos.\frac{1}{2}\omega = \frac{1}{\sqrt{1+tt}},$$

unde colligitur

$$\cos.\omega = \cos.\frac{1}{2}\omega^2 - \sin.\frac{1}{2}\omega^2 = \frac{1-tt}{1+tt}.$$

Erit ergo $b+\cos.\omega = \frac{b+1+(b-1)tt}{1+tt}$ sicque erit

$$\frac{b\partial\omega}{b+\cos.\omega} = \frac{2b\partial t}{(b+1)+(b-1)tt},$$

cuius integratio semper ad arcum circularem reducitur, dummodo fuerit $b>1$.

6. Ad hoc integrale inveniendum notetur esse in genere

$$\int\frac{\partial t}{f+gtt} = \frac{1}{\sqrt{fg}}\,\text{A tang.}\,\frac{t\sqrt{g}}{\sqrt{f}},$$

unde pro nostro casu erit angulus

$$\varphi = \omega - \frac{2b}{\sqrt{bb-1}}\,\text{A tang.}\, t\sqrt{\frac{b-1}{b+1}}.$$

At vero ut horum angulorum differentia geometrice assignari queat, necesse est, ut coefficiens $\frac{2b}{\sqrt{bb-1}}$ sit numerus rationalis; atque adeo iam evidens est, quoties hoc contigerit, semper prodituram esse curvam algebraicam AZ cum circulo proposito arcus aequales habentem.

7. Cum sit $z = b+\cos.\omega$, plures egregiae proprietates huius curvae se offerunt, quas probe notari conveniet; namque si ad Z ducatur tangens ZT et vocetur angulus $CZT=\psi$, erit

$$\sin.\psi = \frac{z\partial\varphi}{\partial\omega};$$

ergo ob $\partial\varphi = \frac{\partial\omega\cos.\omega}{b+\cos.\omega}$ erit $\sin.\psi = \cos.\omega$, ita ut angulus CZT semper

aequetur $90^\circ - \omega$ ideoque ob $AZ = \omega$ semper erit

$$\psi = \frac{\pi}{2} - \omega$$

denotante $\frac{\pi}{2}$ angulum rectum. Hinc, si ex C in tangentem demittatur perpendiculum CT, erit

$$CT = z \sin.\psi = z \cos.\omega = (b + \cos.\omega)\cos.\omega.$$

Posito autem hoc perpendiculo $CT = p$ constat semper esse radium osculi curvae $= \frac{z\partial z}{\partial p}$. Cum igitur sit

$$z\partial z = -\partial\omega \sin.\omega(b + \cos.\omega) \quad \text{et} \quad \partial p = -\partial\omega \sin.\omega(b + 2\cos.\omega),$$

erit radius osculi curvae in Z, quem vocemus r,

$$= \frac{b + \cos.\omega}{b + 2\cos.\omega},$$

qui ergo in initio, ubi $\omega = 0$, erit $r = \frac{b+1}{b+2}$ ideoque minor quam in circulo. At vero pro arcu $\omega = \frac{\pi}{2}$ erit $r = 1$ ideoque radio circuli aequalis. Sumto autem $\omega = \pi$ erit $r = \frac{b-1}{b-2}$. Unde patet, nisi sit $b > 2$, hunc radium osculi fieri negativum sive in plagam contrariam vergere ideoque interea curvam punctum flexus contrarii esse passam, quod eveniet, ubi $\cos.\omega = -\frac{b}{2}$, quod ergo inter $\omega = 90^\circ$ et $\omega = 180^\circ$ cadet. Hocque loco radius osculi erit infinite magnum. Praeterea cum sit $\partial\varphi = \frac{\partial\omega \cos.\omega}{b + \cos.\omega}$, manifestum est curvam supra axem ascendere sive angulum $ACZ = \varphi$ augeri ab $\omega = 0^\circ$ ad $\omega = 90^\circ$, hinc autem istum angulum iterum decrescere atque adeo curvam axem AC secare, antequam fiat $\omega = 180^\circ$, quia tum angulus φ fiet negativus. Quia enim posito $\omega = 180^\circ$ fit $t = \infty$ ideoque $\text{A tang.}\, t\sqrt{\frac{b-1}{b+1}} = 90^\circ$ ideoque $\varphi = 180^\circ\left(1 - \frac{b}{\sqrt{bb-1}}\right)$, ubi $\frac{b}{\sqrt{bb-1}} > 1$.

8. Ex radio osculi invento $r = \frac{b + \cos.\omega}{b + 2\cos.\omega}$ etiam commode assignari potest amplitudo curvae $AZ = \omega$. Si enim amplitudo ponatur ϑ, erit

$$\partial\vartheta = \frac{\partial\omega}{r} = \frac{\partial\omega(b + 2\cos.\omega)}{b + \cos.\omega},$$

hoc est, erit

$$\partial \varphi = \partial \omega + \frac{\partial \omega \cos. \omega}{b + \cos. \omega} = \partial \omega + \partial \varphi$$

sicque amplitudo φ semper aequatur summae angulorum ω et φ, quamdiu scilicet angulus φ supra axem cadit. Si enim infra axem cadat, negative accipi debet. Cum autem amplitudo curvae continuo augeatur, quamdiu curva AZ versus eandem partem est concava, postquam autem coepit in partem contrariam vergere, quod evenit, ubi punctum flexus contrarii datur (iam notavimus tale punctum occurrere, ubi $b + 2\cos. \omega = 0$ seu ubi $\cos. \omega = -\frac{b}{2}$), tum, cum sit $z = b + \cos. \omega$, fiet $z = \frac{1}{2} b$, ita ut punctum flexus contrarii semper incidat in distantiam $CZ = \frac{1}{2} b$; unde colligimus curvam ab initio A, ubi $z = b + 1$, concavitatem axi obvertere, donec fiat distantia $z = \frac{1}{2} b$, et quamdiu distantia minor fuerit quam $\frac{1}{2} b$, concavitatem in partem contrariam vergi, id quod evenire nequit, nisi fuerit $b < 2$, quia $b - 1$ minima distantia curvae a centro C; quamobrem, si fuerit $b > 2$, tota curva nusquam habebit punctum flexus contrarii.

9. Cum autem nostrae curvae algebraicae fieri nequeant, nisi haec formula $\frac{b}{\sqrt{bb-1}}$ aequetur numero rationali, quem ponamus n, hinc vicissim colligitur $b = \frac{n}{\sqrt{nn-1}}$. Tum igitur erit angulus ACZ

$$= \varphi = \omega - 2n \operatorname{A tang.} t \sqrt{\frac{b-1}{b+1}},$$

ubi est $t = \operatorname{tang.} \frac{1}{2}\omega$. Hic igitur erit

$$\frac{b-1}{b+1} = \frac{n - \sqrt{nn-1}}{n + \sqrt{nn-1}} = \frac{1}{(n + \sqrt{nn-1})^2}$$

sicque erit

$$t \sqrt{\frac{b-1}{b+1}} = \frac{t}{n + \sqrt{nn-1}}.$$

Quia igitur necessario sumi debet $n > 1$, manifestum est istam tangentem $t \sqrt{\frac{b-1}{b+1}}$ semper minorem esse quam t. Ponamus ergo brevitatis gratia

$$t \sqrt{\frac{b-1}{b+1}} = u$$

vocemus angulum, cuius tangens est u, $=\theta$; habebimus hanc formulam

$$\varphi = \omega - 2n\theta,$$

de deducitur sequens

CONSTRUCTIO GEOMETRICA CURVARUM QUAESITARUM

10. Monstrabimus igitur, quomodo pro quovis circuli puncto z punctum respondens Z in qualibet curva quaesita definiri queat. Sumto nimirum o n numero quocunque rationali unitate maiore capiatur $b = \frac{n}{\sqrt{nn-1}} = cb$; m vero ex arcu $az = \omega$ habebitur $t = \text{tang.}\frac{1}{2}\omega$ hincque etiam innotescet

$$u = t\sqrt{\frac{b-1}{b+1}} = \frac{t}{n+\sqrt{nn-1}}.$$

unc abscindatur in circulo arcus, cuius tangens est u, qui ponatur $=\theta$, et ia n est numerus rationalis, geometrice assignabitur $=2n\theta$, quo facto nstruatur angulus ACZ aequalis differentiae angulorum ω et $2n\theta$, ut sciet fiat $\varphi = \omega - 2n\theta$, quo facto sumatur distantia $CZ = b + \cos.\omega = bp$, ocque modo pro singulis circuli punctis z determinabuntur puncta correspon- entia Z curvae quaesitae.

11. Hinc patet, quando arcus $az = \omega$ evanescit, tum punctum Z incidere ı ipsum punctum A existente $CA = ba$. At vero sumto arcu $az = 180° = \pi$, uia tum fit $t = \text{tang.}\frac{1}{2}\pi = \infty$, erit etiam $u = \infty$, unde $\theta = 90°$. Pro hoc rgo casu fiet angulus $\varphi = 180° - 2n \cdot 90° = \pi(1-n)$. Quare cum semper sit $\cdot > 1$, angulus φ ad alteram axis partem cadet eritque hic angulus $= \pi(n-1)$.)istantia vero puncti respondentis a centro C erit $b-1$, quae est minima listantia, ad quam nostra curva versus centrum accedere potest. Sufficiet utem hoc modo tractum curvae tantum a distantia maxima $b+1$ usque d minimam $b-1$ descripsisse, propterea quod ultra hos terminos curva ıtrinque aequaliter porrigitur, unde intelligitur tam distantiam maximam uam minimam fore curvae diametros. Denique etiam ultro patet longiudinem curvae a distantia [maxima] ad sequentem minimam semiperipheriae irculi propositi aequari. Et quia angulus inter maximam et minimam listantiam, qui est $(n-1)\pi$, cum peripheria circuli est commensurabilis, equitur numerum diametrorum semper esse debere finitum.

12. Hinc etiam intelligitur, quomodo aequationem inter coordinatas $CP = x$ et $PZ = y$ erui oporteat; cum enim sit

$$\text{tang.}\, \varphi = \frac{y}{x} \quad \text{et} \quad \text{tang.}\, \frac{1}{2}\varphi = \sqrt{\frac{z-x}{z+x}},$$

cui aequari debet tang. $\left(\frac{1}{2}\omega - n\theta\right)$. Quia vero posuimus tang. $\frac{1}{2}\omega = t$, erit $\cos.\omega = \frac{1-tt}{1+tt}$, unde ob $z = b + \frac{1-tt}{1+tt}$ elicitur $tt = \frac{b+1-z}{z-b+1}$ hincque

$$uu = \frac{b-1}{b+1}tt = \frac{bb-1-(b-1)z}{(b+1)z-bb+1}$$

sicque t et ω per functiones ipsius z ideoque etiam tang. $n\theta$ per talem functionem exprimetur, unde etiam tangens anguli $\frac{1}{2}\omega - n\theta$ per functionem solius z definietur. Hinc sumtis quadratis formula $\frac{z-x}{z+x}$ aequatur functioni rationali ipsius z, quae aequatio denique ob $z = \sqrt{xx+yy}$ sumendis quadratis ad aequationem rationalem inter x et y reducitur, quae autem plerumque ad plurimas dimensiones assurgit, siquidem pro casu simplicissimo, quo $n = 2$, ad sextum ordinem ascendit.

DESCRIPTIO CURVAE SIMPLICISSIMAE QUO $n = 2$

13. Hic ergo ob $n = 2$ erit $b = \frac{2}{\sqrt{3}} = \sec. 30°$ ideoque proxime $b = 1{,}1547$. Maxima igitur curvae distantia a centro C (Fig. 3) seu quasi absis summa erit $CA = b + 1 = 2{,}1547$, ad quam curva est normalis, ibique radius osculi erit $r = \frac{b+1}{b+2} = 0{,}6830$. Minima distantia erit $b - 1 = 0{,}1547$, quae a maxima distabit angulo $= 180°$ ideoque in axem AC continuatum cadet, quae sit CI, ubi curva iterum ad axem erit normalis. At vero radius osculi in I erit $\frac{b-1}{b-2} = -0{,}1830$. Longitudo autem curvae ab abside summa A ad imam I protensae aequabitur semiperipheriae circuli radio 1 descripti.

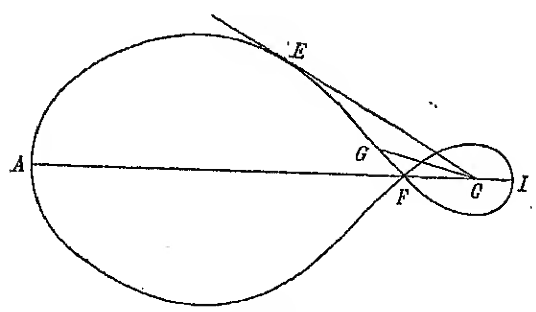

Fig. 3.

14. Pro aliis curvae punctis memorabilibus definiendis sumto arcu $Z = \omega$ erit distantia $CZ = b + \cos\omega$. Pro angulo autem $ACZ = \varphi$ habemus $\text{tang.}\,\frac{1}{2}\varphi = \text{tang.}\left(\frac{1}{2}\omega - 2\theta\right)$, ubi posito $\text{tang.}\,\frac{1}{2}\omega = t$ erit

$$\text{tang.}\,\theta = u = t\sqrt{\frac{b-1}{b+1}} = 0{,}2679\,t$$

vicissim

$$t = u\sqrt{\frac{b+1}{b-1}} = 3{,}7321\,u.$$

m igitur sit $\text{tang.}\,\theta = u$, erit $\text{tang.}\,2\theta = \frac{2u}{1-uu}$, unde fit

$$\text{tang.}\left(\frac{1}{2}\omega - 2\theta\right) = \frac{t(1-uu)-2u}{1-uu+2tu} = \text{tang.}\,\frac{1}{2}\varphi.$$

15. Sumamus nunc arcum $AE = 90^\circ = \frac{1}{2}\pi$ eritque distantia $CE = b$; angulus $\psi = 90^\circ - \omega = 0$, unde patet rectam CE curvam E tangere ibique dium osculi fore $= 1$. Pro angulo ACE investigando habemus $t = 1$ et $= 0{,}2679 = \text{tang.}\,\theta$. Erit ergo angulus $\theta = 15^\circ\,0'$ ideoque $\frac{1}{2}\varphi = 15^\circ\,0'$ hocque odo erit angulus $ACE = 30^\circ$.

16. Hinc igitur curva ad axem appropinquabit eamque mox secabit in ?, ubi ergo, cum fiat $\varphi = 0$, erit $t(1-uu) = 2u$ sive $3{,}7321\,(1-uu) = 2$, nde reperitur $uu = 0{,}4641$ hincque $t = 2{,}7321$.[1]) Erit ergo $\frac{1}{2}\omega = 69^\circ\,54'$ ideoque $= 139^\circ\,48'$. Unde patet curvam hic ad axem sub angulo $49^\circ\,48'$ esse inclinatam, distantiam vero fore $CF = b - \sin.\,49^\circ\,48' = 0{,}3909$. Radius osculi hoc oco erit $= -1{,}0483$. Hic ergo curva iam in contrariam partem est inflexa deoque punctum flexus contrarii praecessit punctum F.

17. Ad hoc ergo punctum, quod sit in G, inveniendum iam supra otavimus id incidere, ubi distantia $CG = \frac{1}{2}b = 0{,}5773$, ita ut $\cos.\,\omega = -\frac{1}{2}b$ deoque $\omega = 125^\circ\,16'$. Quare hoc loco curva ad rectam CG inclinatur sub ngulo $35^\circ\,16'$. Quia porro est $\frac{1}{2}\omega = 62^\circ\,38'$, erit $t = 1{,}9319$ hincque porro $u = 0{,}5176$, quae est tangens anguli θ, qui consequenter erit $27^\circ\,22'$, ergo

1) Pro hoc falso numero substituendum est $t = 2{,}5425$, ex quo valore sequitur $\frac{1}{2}\omega = 68^\circ 32'$. Quamobrem etiam numeri sequentes corrigendi sunt. A. K.

$\frac{1}{2}\varphi = 7^\circ 54'$, consequenter angulus $FCG = 15^\circ 48'$. Ex his autem principalibus curvae punctis tractus curvae facile satis exacte describi poterit, unde, cum recta AI simul curvae sit diameter, tota curva habet hanc figuram (Fig. 3).

SUPPLEMENTUM

18. Solutio sequentis problematis non parum elegantis omnes curvas methodo praecedente inventas multo facilius et commodius largietur.

PROBLEMA

Invenire curvam EZ (Fig. 4) ad punctum fixum C relatam, cuius quilibet arcus EZ ad angulum EZC ubique eundem teneat rationem.

SOLUTIO

19. Hic igitur statim patet arcum curvae EZ, quia angulo EZC est proportionalis, aequalem fore arcui circulari eundem angulum metientis ideoque, si hae curvae fuerint algebraicae, eas scopo nostro esse satisfacturas.

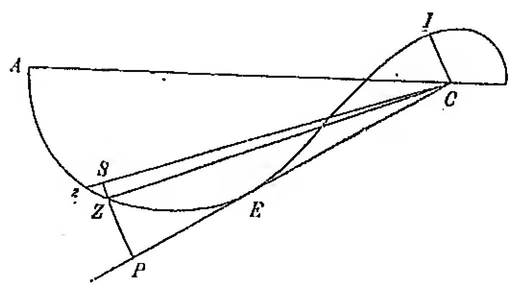

Fig. 4.

Ad eas inveniendas ponamus angulum $ECZ = \varphi$ et distantiam $CZ = z$, ut habeamus pro situ proximo $ZS = z\partial\varphi$ et $zS = \partial z$. Ponamus nunc angulum $EZC = \omega$, arcum vero $EZ = a\omega$, et quia omnes curvae similes ad idem punctum C relatae aeque satisfaciunt, sumere licebit $a = 1$, ut sit arcus $EZ = \omega$ eiusque ergo elementum $Zz = \partial\omega$, et nunc triangulum ZzS statim praebet has duas aequationes

$$\partial z = \partial\omega \cos.\omega \quad \text{et} \quad z\partial\varphi = \partial\omega \sin.\omega.$$

20. Prior harum aequationum integrata statim dat $z = b + \sin.\omega$, unde ex altera fit $\partial\varphi = \frac{\partial\omega \sin.\omega}{b + \sin.\omega}$. Hinc statim manifestum est in puncto E, ubi arcus EZ evanescit, fore etiam angulum $\omega = 0$ ideoque distantiam $CE = b$ et hanc rectam CE fore curvae tangentem in ipso initio E.

21. Pro elemento ergo angulari $\partial\varphi$ habemus

$$\partial\varphi = \partial\omega - \frac{b\partial\omega}{b+\sin.\omega} \quad \text{ideoque} \quad \varphi = \omega - \int \frac{b\partial\omega}{b+\sin.\omega},$$

ad quam formulam integrandam ponamus $\operatorname{tang.} \frac{1}{2}\omega = t$, unde fit $\sin.\omega = \frac{2t}{1+tt}$ et $\partial\omega = \frac{2\partial t}{1+tt}$, unde oritur formula

$$\frac{b\partial\omega}{b+\sin.\omega} = \frac{2b\partial t}{b(1+tt)+2t}.$$

Ponamus $\frac{1}{b} = \cos.\beta$, ut oriatur

$$\frac{b\partial\omega}{b+\sin.\omega} = \frac{2\partial t}{1+tt+2t\cos.\beta},$$

cuius formulae integrale semper exprimet arcum circuli, si modo fuerit $b > 1$ et $\frac{1}{b}$ per cosinum cuiuspiam anguli referri queat. Constat autem huius formulae integrale fore

$$= \frac{2}{\sin.\beta} \operatorname{A tang.} \frac{t\sin.\beta}{1+t\cos.\beta},$$

ita ut iam nacti simus hanc aequationem

$$\varphi = \omega - \frac{2}{\sin.\beta} \operatorname{A tang.} \frac{t\sin.\beta}{1+t\cos.\beta};$$

unde patet, quoties $\sin.\beta$ fuerit numerus rationalis, istum angulum semper geometrice assignari posse ideoque curvam nostram fore algebraicam, et quia angulum β infinitis modis accipere licet, simul reperiri innumerabiles curvas algebraicas scopo nostro satisfacientes, quippe quarum omnes arcus per arcus circulares mensurantur. Evidens autem est has curvas cum iis, quas ante invenimus, perfecte convenire, quia hic tantum aliud principium est assumtum in E.

22. Quoniam igitur $\sin.\beta$ debet esse numerus rationalis, ponamus $\frac{1}{\sin.\beta} = n$, ita ut n sit numerus quicunque unitate maior sive integer sive fractus, ac posito brevitatis gratia

$$\operatorname{A tang.} \frac{t\sin.\beta}{1+t\cos.\beta} = \theta$$

erit

$$\varphi = \omega - 2n\theta,$$

qui ergo angulus in principio, ubi $\omega = 0$, etiam evanescit. Erit igitur $\frac{1}{2}\varphi = \frac{1}{2}\omega - n\theta$ ac positis coordinatis orthogonalibus $CP = x$ et $PZ = y$ erit

$$\text{tang.}\,\varphi = \frac{y}{x} \quad \text{et} \quad \text{tang.}\,\frac{1}{2}\varphi = \frac{y}{z+x} = \sqrt{\frac{z-x}{z+x}}.$$

Cum porro sit

$$z = b + \sin.\omega = b + \frac{2t}{1+tt},$$

patet etiam t aequari functioni ipsius z hincque etiam tang. θ, ita ut hinc pro quovis casu aequatio inter coordinatas orthogonales x et y erui queat.

23. Investigemus nunc praecipua puncta huius curvae ac primo quidem capiamus arcum $EA = 90^\circ = \frac{\pi}{2}$ eritque angulus ω rectus et distantia CA ad curvam erit normalis simulque erit curvae diameter, circa quam curva utrinque pari tractu protenditur. Hic igitur erit tang. $\frac{1}{2}\omega = t = 1$ ideoque

$$\text{tang.}\,\theta = \frac{\sin.\beta}{1+\cos.\beta} = \text{tang.}\,\frac{1}{2}\beta,$$

ita ut $\theta = \frac{1}{2}\beta$, unde invento hoc angulo β, cuius cosinus est $\frac{1}{b}$, erit angulus $ECA = \frac{\pi}{2} - n\beta$. Ipsa autem distantia CA erit $b+1$, quae erit maxima, ad quam curva pertingere potest.

24. Consideremus nunc portionem huius curvae a puncto E retro protensam ac sumamus arcum EI quadranti aequalem, unde statui oportebit $\omega = -\frac{\pi}{2}$, atque in hoc puncto I erit distantia $CI = b - 1$, quae est omnium minima, ad quam curva descendere potest, hicque iterum erit CI ad curvam normalis pariterque eius diameter, unde sufficiet curvam tantum ab A per E usque ad I descripsisse.

25. Hoc igitur casu ob $t = -1$ erit $\theta = \text{A tang.}\,\frac{-\sin.\beta}{1-\cos.\beta}$ sicque iste angulus θ erit negativus eiusque tangens $\frac{\sin.\beta}{1-\cos.\beta}$, quae expressio est cotangens anguli $\frac{1}{2}\beta$, sicque erit $-\theta = \frac{\pi}{2} - \frac{1}{2}\beta$, unde prodit angulus

$$ECI = \varphi = -\frac{\pi}{2} + 2n\left(\frac{\pi}{2} - \frac{1}{2}\beta\right) = \left(n - \frac{1}{2}\right)\pi - n\beta,$$

quamobrem angulus inter distantiam maximam $CA = b + 1$ et minimam $CI = b - 1$ interceptus erit $ACI = (n - 1)\pi$, prorsus uti supra est inventus.

26. Consideremus denique casum, quo arcus EZ semiperipheriae aequalis accipitur sive $\omega = \pi$, ubi ergo distantia curvam iterum tanget; tum igitur erit $t = \infty$ et $\text{tang.}\,\theta = \text{tang.}\,\beta$ ideoque $\theta = \beta$ sicque erit angulus $\varphi = \pi - 2n\beta$, qui est duplo maior quam angulus ECA, prorsus ut indoles diametri postulat. Ceterum hic notasse iuvabit omnes formulas hic inventas ad praecedentes reduci posse, si loco t scribatur $\frac{1-t}{1+t}$ simulque angulus φ minuatur angulo $ECA = \frac{\pi}{2} - n\beta$.

DE LINEIS CURVIS QUARUM RECTIFICATIO PER DATAM QUADRATURAM MENSURATUR

Commentatio 817 indicis ENESTROEMIANI
Opera postuma 1, Petropoli 1862, p. 439—451

1. Satis notum est problema inter Geometras olim multum agitatum, quo lineae curvae quaerebantur, quarum rectificatio a datae curvae quadratura pendeat, cuius solutionem etiam HERMANNUS[1]) beatae memoriae contra exspectationem summorum Geometrarum ita feliciter expedivit, ut non solum infinitas curvas algebraicas, quarum rectificatio a data quadratura penderet, exhibuerit, sed etiam hanc conditionem adiunxerit, ut istae curvae unum duosve atque adeo tot, quot lubuerit, haberent arcus absolute rectificabiles. Cum autem methodus, qua HERMANNUS erat usus, nimis videretur recondita neque ad uberiorem usum in Analysi satis accommodata, aliam methodum planam ac facilem investigavi, cuius ope non solum hoc problema, sed etiam omnia, quae huius generis occurrere queant, expedite resolvi possunt. Complectitur ista methodus quasi novam Analyseos speciem, cuius elementa, quae multo latius patere videntur, dilucide exposui in Novis Commentariis Academiae imperialis Petropolitanae.[2])

2. Huius methodi beneficio, si proponatur quadratura seu formula integralis quaecunque $\int Z dz$ existente Z functione ipsius z quacunque, innumerabiles curvae algebraicae definiri possunt, quarum rectificatio ab ista formula ita pendeat, ut eius integratione concessa omnes harum curvarum arcus in-

1) Vide notam p. 82. A. K.
2) Vide notam p. 248. A. K.

definite definiri queant. Per variabilem scilicet z eiusmodi expressiones algebraicae pro coordinatis x et y assignantur, ut inde formulae $\sqrt{(dx^2 + dy^2)}$ integratio perducatur ad huiusmodi formam $\alpha\int Zdz + V$, ubi V sit functio algebraica ipsius z. Verum haec quantitas V non arbitrio nostro relinquitur, etiamsi infinitis modis variari queat; atque hinc ope methodi a me traditae problema non ita resolvi potest, ut curvarum inveniendarum arcus absolute per formulam propositam $\int Zdz$ eiusve multiplum $\alpha\int Zdz$ exprimantur.

3. Maxime igitur diversum est problema, quo quaeruntur curvae algebraicae, quarum arcus per propositam quampiam formulam integralem $\int Zdz$ simpliciter sine adiunctione cuiusdam functionis algebraicae exprimantur. Atque adeo hoc problema saepenumero ne solutionem quidem admittere videtur. Ita si sit $Z = \frac{a}{z}$ et curva algebraica sit investiganda, cuius arcus per $a\int\frac{dz}{z}$ seu alz exprimatur, vehementer dubito, num quisquam unquam huiusmodi curvam sit reperturus. Quaestio scilicet huc redit, ut eiusmodi binae functiones algebraicae ipsius z inveniantur, quae pro coordinatis x et y substitutae praebeant $\sqrt{(dx^2 + dy^2)} = \frac{adz}{z}$. Postquam equidem hoc problema multis modis tentavi aliisque insignibus Geometris enodandum proposui,[1]) neque ego neque quisquam alius solutionem assequi potuimus, cum tamen, in genere si quaeratur curva algebraica, cuius rectificatio a logarithmis pendeat, problema sit facillimum atque adeo parabola conica ei satisfaciat. Unde concludendum est hoc problema vel omnino nullam solutionem admittere vel methodum adhuc plane nobis incognitam requirere.

4. Evenire quoque posse videtur, ut huiusmodi problemata unicam tantum solutionem admittant neque plus una curva exhiberi queat, cuius arcus per datam formulam integralem exprimantur. Equidem hoc sum expertus in formula $\int\frac{adz}{\sqrt{(aa - zz)}}$, qua arcus circuli exprimitur; nullam enim aliam lineam curvam algebraicam invenire potui,[2]) cuius arcus per eandem formulam exprimeretur. Sic nulla videtur extare curva algebraica, cuius arcui cuicunque aequalis arcus circularis exhiberi queat, etiamsi innumerabiles lineae algebraicae sint notae, quarum rectificatio a rectificatione circuli pendeat. Statim enim atque hae curvae a circulo sunt diversae, earum arcus aequantur aggregato ex arcu quodam circulari et linea geometrice assignabili,

1) Vide p. 88. 2) Vide p. 83. A. K.

quae nonnisi certis casibus in nihilum abire potest. Idem tenendum est de formulis $\int \frac{a dz}{\sqrt{(2az - zz)}}$ et $\int \frac{aadz}{aa + zz}$ aliisque, in quas illa formula $\int \frac{a dz}{\sqrt{(aa - zz)}}$ per substitutiones transformari potest.

5. Dantur tamen etiam eiusmodi formulae $\int Zdz$, pro quibus innumerabiles curvae algebraicae exhiberi possunt, ita ut infinitae curvae algebraicae assignari queant, in quarum una si capiatur arcus quicunque, in reliquis omnibus pares arcus abscindere liceat. Huc imprimis pertinet problema olim a Celebb. BERNOULLIIS[1]) tractatum, quo curva algebraica quaerebatur, cuius rectificatio cum rectificatione curvae elasticae conveniret seu per hanc formulam $\int \frac{aadz}{\sqrt{(a^4 - z^4)}}$ exprimeretur; invenerunt enim lineam quarti ordinis, ob figuram *lemniscatam* dictam, quae huic scopo satisfaceret. Ostendam autem praeter lemniscatam infinitas alias exhiberi posse curvas algebraicas, quarum arcus generatim per eandem formulam exprimantur. Cum igitur lemniscata docente Ill. FAGNANO[2]) hanc habeat insignem proprietatem, ut in ea perinde atque in circulo arcus quotcunque aequales abscindi queant, eadem proprietas quoque in omnes curvas, quarum arcus per eandem formulam $\int \frac{aadz}{\sqrt{(a^4 - z^4)}}$ exprimuntur, competet; quae ergo merentur, ut diligentius evolvantur.

6. Methodus quidem, qua hanc investigationem suscipio, per se satis est plana et ope calculi angulorum facile expediri potest. Si enim arcus cuiuspiam curvae per hanc formulam $\int Zdz$ debeat exprimi, vocatis coordinatis orthogonalibus x et y atque introducto angulo quocunque φ statuatur

$$dx = Zdz \cos.\varphi \quad \text{et} \quad dy = Zdz \sin.\varphi;$$

sic enim prodibit arcus elementum

$$\sqrt{(dx^2 + dy^2)} = Zdz \quad \text{ipseque arcus} = \int Zdz.$$

1) IAC. BERNOULLI, *Solutio problematis* LEIBNITIANI *de curva accessus et recessus aequabilis a puncto dato, mediante rectificatione curvae elasticae*, Acta erud. 1694, p. 276; *Opera*, p. 601. *Constructio curvae accessus et recessus aequabilis, ope rectificationis curvae cuiusdam algebraicae*, Acta erud. 1694, p. 336; *Opera* p. 608.

IOH. BERNOULLI, *Constructio facilis curvae accessus aequabilis a puncto dato per rectificationem curvae algebraicae*, Acta erud. 1694, p. 394; *Opera omnia*, t. 1, p. 119. A. K.

2) G. C. FAGNANO, *Produzioni matematiche*, t. 2, Pesaro 1750; *Opere matematiche*, t. 2, Milano-Roma-Napoli 1911. A. K.

Unde quaestio huc redit, ut, quemadmodum arcus φ ad variabilem z comparatus esse debeat, investigetur, ut ambae formulae $Zdz\cos.\varphi$ et $Zdz\sin.\varphi$ evadant integrabiles; quippe quod conditio, qua curvae debent esse algebraicae, postulat. Hunc in finem illae integrationes per solos sinus et cosinus angulorum sunt absolvendae neque ipsi anguli, qui formulas redderent transcendentes, sunt admittendi.

DE CURVA LEMNISCATA

7. Propositum ergo sit curvas algebraicas investigare, quarum arcus indefinite per hanc formulam integralem $\int\frac{aadz}{\sqrt{(a^4-z^4)}}$ exprimantur, et positis coordinatis orthogonalibus x et y statuamus

$$dx=\frac{aadz}{\sqrt{(a^4-z^4)}}\cos.\varphi \quad \text{et} \quad dy=\frac{aadz}{\sqrt{(a^4-z^4)}}\sin.\varphi,$$

quas formulas absolute integrabiles reddi oportet. Ut partem $\frac{aadz}{\sqrt{(a^4-z^4)}}$ quoque ad calculum angulorum perducam, pono

$$zz=aa\sin.\theta,$$

ut fiat

$$\sqrt{(a^4-z^4)}=aa\cos.\theta,$$

et ob $z=a\sqrt{\sin.\theta}$ erit

$$dz=\frac{ad\theta\cos.\theta}{2\sqrt{\sin.\theta}} \quad \text{et} \quad \frac{aadz}{\sqrt{(a^4-z^4)}}=\frac{ad\theta}{2\sqrt{\sin.\theta}}.$$

Hinc itaque nostrae formulae integrabiles reddendae sunt

$$dx=\frac{ad\theta\cos.\varphi}{2\sqrt{\sin.\theta}} \quad \text{et} \quad dy=\frac{ad\theta\sin.\varphi}{2\sqrt{\sin.\theta}}.$$

Ponamus ergo $\varphi=n\theta$, ut sit

$$\frac{2dx}{a}=\frac{d\theta\cos.n\theta}{\sqrt{\sin.\theta}} \quad \text{et} \quad \frac{2dy}{a}=\frac{d\theta\sin.n\theta}{\sqrt{\sin.\theta}},$$

et videamus, quinam valores pro n sumti has ambas formulas integrabiles reddant.

8. Consideremus in genere has formulas

$$\frac{d\theta \cos. m\theta}{\sqrt{\sin. \theta}} \quad \text{et} \quad \frac{d\theta \sin. m\theta}{\sqrt{\sin. \theta}}$$

et perpendamus, quomodo ad simpliciores revocari possint. Talis enim reductio unica via esse videtur ad casus integrabilitatis eruendos. Statuamus ergo primo

$$P = \cos. (m-1)\theta \sqrt{\sin. \theta}$$

et differentiando habebitur

$$dP = \frac{-(m-1)\, d\theta \sin. (m-1)\theta \sin. \theta + \frac{1}{2}\, d\theta \cos. (m-1)\theta \cos. \theta}{\sqrt{\sin. \theta}}.$$

Cum autem sit

$$\sin. \alpha\theta \sin. \theta = \frac{1}{2} \cos. (\alpha - 1)\theta - \frac{1}{2} \cos. (\alpha + 1)\theta$$

et

$$\cos. \alpha\theta \cos. \theta = \frac{1}{2} \cos. (\alpha - 1)\theta + \frac{1}{2} \cos. (\alpha + 1)\theta,$$

erit

$$dP = \frac{-(2m-3)\, d\theta \cos. (m-2)\theta + (2m-1)\, d\theta \cos. m\theta}{4\sqrt{\sin. \theta}},$$

unde obtinetur

$$\int \frac{d\theta \cos. m\theta}{\sqrt{\sin. \theta}} = \frac{4 \cos. (m-1)\theta \sqrt{\sin. \theta}}{2m-1} + \frac{2m-3}{2m-1} \int \frac{d\theta \cos. (m-2)\theta}{\sqrt{\sin. \theta}}.$$

9. Si deinde simili modo statuamus

$$Q = \sin. (m-1)\theta \sqrt{\sin. \theta},$$

erit differentiando

$$dQ = \frac{(m-1)\, d\theta \cos. (m-1)\theta \sin. \theta + \frac{1}{2}\, d\theta \sin. (m-1)\theta \cos. \theta}{\sqrt{\sin. \theta}}.$$

Cum vero sit

$$\cos. \alpha\theta \sin. \theta = -\frac{1}{2} \sin. (\alpha - 1)\theta + \frac{1}{2} \sin. (\alpha + 1)\theta$$

et

$$\sin. \alpha\theta \cos. \theta = +\frac{1}{2} \sin. (\alpha - 1)\theta + \frac{1}{2} \sin. (\alpha + 1)\theta,$$

erit per has substitutiones

$$dQ = \frac{-(2m-3)\,d\theta \sin.(m-2)\theta + (2m-1)\,d\theta \sin.m\theta}{4\sqrt{\sin.\theta}}.$$

Unde singulis partibus integratis consequemur

$$\int \frac{d\theta \sin.m\theta}{\sqrt{\sin.\theta}} = \frac{4\sin.(m-1)\theta\sqrt{\sin.\theta}}{2m-1} + \frac{2m-3}{2m-1}\int \frac{d\theta \sin.(m-2)\theta}{\sqrt{\sin.\theta}}$$

hincque ergo patet, si formulae propositae $\frac{d\theta \cos.n\theta}{\sqrt{\sin.\theta}}$ et $\frac{d\theta \sin.n\theta}{\sqrt{\sin.\theta}}$ fuerint integrabiles casu $n=\lambda$, tum etiam integrabiles esse futuras casibus $n=\lambda+2$, $n=\lambda+4$, $n=\lambda+6$ etc. sicque ex uno infinitos resultare casus integrabiles.

10. Ex his autem reductionibus statim unus se offert casus absolute integrabilis, scilicet quando $2m-3=0$ seu $m=\frac{3}{2}$; unde obtinemus

$$\int \frac{d\theta}{\sqrt{\sin.\theta}} \cos.\frac{3}{2}\theta = 2\cos.\frac{1}{2}\theta\sqrt{\sin.\theta}$$

et

$$\int \frac{d\theta}{\sqrt{\sin.\theta}} \sin.\frac{3}{2}\theta = 2\sin.\frac{1}{2}\theta\sqrt{\sin.\theta}.$$

Deinde integratio succedet casu $m=\frac{7}{2}$ seu $2m=7$, unde fit

$$\int \frac{d\theta}{\sqrt{\sin.\theta}} \cos.\frac{7}{2}\theta = \frac{2}{3}\cos.\frac{5}{2}\theta\sqrt{\sin.\theta} + 2\cdot\frac{2}{3}\cos.\frac{1}{2}\theta\sqrt{\sin.\theta},$$

$$\int \frac{d\theta}{\sqrt{\sin.\theta}} \sin.\frac{7}{2}\theta = \frac{2}{3}\sin.\frac{5}{2}\theta\sqrt{\sin.\theta} + 2\cdot\frac{2}{3}\sin.\frac{1}{2}\theta\sqrt{\sin.\theta}.$$

Hinc progressus patet ad casum $m=\frac{11}{2}$ seu $2m=11$, qui dat

$$\int \frac{d\theta}{\sqrt{\sin.\theta}} \cos.\frac{11}{2}\theta = \frac{2}{5}\cos.\frac{9}{2}\theta\sqrt{\sin.\theta} + \frac{2\cdot4}{3\cdot5}\cos.\frac{5}{2}\theta\sqrt{\sin.\theta} + 2\cdot\frac{2\cdot4}{3\cdot5}\cos.\frac{1}{2}\theta\sqrt{\sin.\theta},$$

$$\int \frac{d\theta}{\sqrt{\sin.\theta}} \sin.\frac{11}{2}\theta = \frac{2}{5}\sin.\frac{9}{2}\theta\sqrt{\sin.\theta} + \frac{2\cdot4}{3\cdot5}\sin.\frac{5}{2}\theta\sqrt{\sin.\theta} + 2\cdot\frac{2\cdot4}{3\cdot5}\sin.\frac{1}{2}\theta\sqrt{\sin.\theta},$$

et sequens casus $m=\frac{15}{2}$ praebebit

$$\int\frac{d\theta}{\sqrt{\sin.\theta}}\cos.\frac{15}{2}\theta=\left(\frac{2}{7}\cos.\frac{13}{2}\theta+\frac{2\cdot6}{5\cdot7}\cos.\frac{9}{2}\theta+\frac{2\cdot4\cdot6}{3\cdot5\cdot7}\cos.\frac{5}{2}\theta+2\cdot\frac{2\cdot4\cdot6}{3\cdot5\cdot7}\cos.\frac{1}{2}\theta\right)\sqrt{\sin.\theta},$$

$$\int\frac{d\theta}{\sqrt{\sin.\theta}}\sin.\frac{15}{2}\theta=\left(\frac{2}{7}\sin.\frac{13}{2}\theta+\frac{2\cdot6}{5\cdot7}\sin.\frac{9}{2}\theta+\frac{2\cdot4\cdot6}{3\cdot5\cdot7}\sin.\frac{5}{2}\theta+2\cdot\frac{2\cdot4\cdot6}{3\cdot5\cdot7}\sin.\frac{1}{2}\theta\right)\sqrt{\sin.\theta}.$$

11. Ut in coefficientibus angulorum fractiones evitemus, ponamus $\theta=2\omega$, ut sit

$$zz=aa\sin.2\omega \quad \text{seu} \quad \sin.2\omega=\frac{zz}{aa},$$

unde erit

$$\sin.\omega=\frac{1}{2}\sqrt{\left(1+\frac{zz}{aa}\right)}-\frac{1}{2}\sqrt{\left(1-\frac{zz}{aa}\right)}$$

et

$$\cos.\omega=\frac{1}{2}\sqrt{\left(1+\frac{zz}{aa}\right)}+\frac{1}{2}\sqrt{\left(1-\frac{zz}{aa}\right)}.$$

Atque infinitas curvas algebraicas exhibere poterimus, quarum arcus seu valor integralis

$$\int\sqrt{(dx^2+dy^2)}$$

praecise fiat aequalis formulae

$$\int\frac{aadz}{\sqrt{(a^4-z^4)}}=a\int\frac{d\omega}{\sqrt{\sin.2\omega}}.$$

Ac curva quidem prima eaque simplicissima his continebitur coordinatis

$$x=a\cos.\omega\sqrt{\sin.2\omega} \quad \text{et} \quad y=a\sin.\omega\sqrt{\sin.2\omega},$$

ex quibus fit

$$xx+yy=aa\sin.2\omega \quad \text{et} \quad \sqrt{(xx+yy)}=a\sqrt{\sin.2\omega}.$$

Hinc ergo porro elicitur

$$\cos.\omega=\frac{x}{\sqrt{(xx+yy)}} \quad \text{et} \quad \sin.\omega=\frac{y}{\sqrt{(xx+yy)}}$$

ideoque

$$\sin.2\omega=2\sin.\omega\cos.\omega=\frac{2xy}{xx+yy}.$$

Quo valore substituto habebitur aequatio inter solas x et y pro curva

$$(xx+yy)^2 = 2aaxy,$$

quae est ipsa aequatio lemniscatae.

12. Secunda curva algebraica, cuius arcus per eandem formulam

$$\int \frac{aadz}{\sqrt{(a^4-z^4)}} = a\int \frac{d\omega}{\sqrt{\sin. 2\omega}}$$

exprimuntur, continebitur his coordinatis

$$x = \frac{a}{3}(\cos. 5\omega + 2\cos. \omega)\sqrt{\sin. 2\omega},$$

$$y = \frac{a}{3}(\sin. 5\omega + 2\sin. \omega)\sqrt{\sin. 2\omega}.$$

Tertia porro curva aeque satisfaciens his

$$x = \frac{a}{5}\left(\cos. 9\omega + \frac{4}{3}\cos. 5\omega + \frac{4\cdot 2}{3\cdot 1}\cos. \omega\right)\sqrt{\sin. 2\omega},$$

$$y = \frac{a}{5}\left(\sin. 9\omega + \frac{4}{3}\sin. 5\omega + \frac{4\cdot 2}{3\cdot 1}\sin. \omega\right)\sqrt{\sin. 2\omega}.$$

Quarta vero his

$$x = \frac{a}{7}(\cos. 13\omega + \frac{6}{5}\cos. 9\omega + \frac{6\cdot 4}{5\cdot 3}\cos. 5\omega + \frac{6\cdot 4\cdot 2}{5\cdot 3\cdot 1}\cos. \omega)\sqrt{\sin. 2\omega},$$

$$y = \frac{a}{7}(\sin. 13\omega + \frac{6}{5}\sin. 9\omega + \frac{6\cdot 4}{5\cdot 3}\sin. 5\omega + \frac{6\cdot 4\cdot 2}{5\cdot 3\cdot 1}\sin. \omega)\sqrt{\sin. 2\omega}.$$

Quinta hinc sponte formari potest

$$x = \frac{a}{9}(\cos. 17\omega + \frac{8}{7}\cos. 13\omega + \frac{8\cdot 6}{7\cdot 5}\cos. 9\omega + \frac{8\cdot 6\cdot 4}{7\cdot 5\cdot 3}\cos. 5\omega + \frac{8\cdot 6\cdot 4\cdot 2}{7\cdot 5\cdot 3\cdot 1}\cos. \omega)\sqrt{\sin. 2\omega},$$

$$y = \frac{a}{9}(\sin. 17\omega + \frac{8}{7}\sin. 13\omega + \frac{8\cdot 6}{7\cdot 5}\sin. 9\omega + \frac{8\cdot 6\cdot 4}{7\cdot 5\cdot 3}\sin. 5\omega + \frac{8\cdot 6\cdot 4\cdot 2}{7\cdot 5\cdot 3\cdot 1}\sin. \omega)\sqrt{\sin. 2\omega}.$$

13. Sic igitur infinitas nacti sumus curvas algebraicas, quarum rectificatio plane congruit cum rectificatione lemniscatae, ita ut cuique arcui huius

curvae in omnibus illis arcus aequales abscindi possint; vix tamen asseverare ausim praeter has nullas dari alias curvas algebraicas, quae eadem praeditae sint proprietate. Methodus enim, qua sum usus, non ita est comparata, ut pro generali haberi possit, propterea quod in formulis § 7 angulum φ tanquam multiplum anguli θ spectavi, cum tamen fortasse alia relatio inter eos intercedere possit, quae ad integrationem aeque sit accommodata. Hoc inde suspicari licet, quod, si aliae formulae integrales $\int Zdz$ proponantur eaeque pari modo ad angulum quempiam θ reducantur, integratio non succedat pro angulo φ multiplum anguli θ assumendo, cum tamen saepenumero aliae relationes negotium conficiant. Huiusmodi casus probe notasse iuvabit, quoniam inde forte methodum latius patentem talia problemata tractandi derivare licebit, si cunctae operationes, quas varia problemata singularia requirunt, diligenter perpendantur atque inter se conferantur. Quem in finem unam atque alteram solutionem similium quaestionum adiungam.

DE PARABOLA

14. Propositum itaque sit alias curvas algebraicas investigare, quarum rectificatio conveniat cum rectificatione parabolae seu quarum arcus indefinite exprimatur per hanc formulam

$$\int \frac{dz}{a} \sqrt{(aa+zz)}.$$

Necesse igitur est, ut coordinatae orthogonales ita se habeant

$$x = \int \frac{dz \cos.\varphi}{a} \sqrt{(aa+zz)} \quad \text{et} \quad y = \int \frac{dz \sin.\varphi}{a} \sqrt{(aa+zz)},$$

ubi definiendum erit, qualem relationem angulus φ ad variabilem z tenere debeat, ut ambae istae formulae integrabiles reddantur. Ponamus ergo

$$\frac{z}{a} = \operatorname{tang.}\theta,$$

ut fiat

$$\sqrt{(aa+zz)} = a \sec.\theta = \frac{a}{\cos.\theta},$$

et cum sit

$$\frac{dz}{a} = \frac{d\theta}{\cos.\theta^2},$$

erit arcus

$$\int \frac{dz}{a} \sqrt{(aa+zz)} = \int \frac{a\,d\theta}{\cos.\theta^3}$$

et coordinatae

$$x = a\int \frac{d\theta \cos.\varphi}{\cos.\theta^3} \quad \text{et} \quad y = a\int \frac{d\theta \sin.\varphi}{\cos.\theta^3}$$

atque hic iterum observo certa multipla anguli θ pro angulo φ exhiberi posse, quibus ambae formulae integrabiles evadant. Statuatur ergo $\varphi = n\theta$, ut habeamus pro coordinatis sequentes expressiones

$$x = a\int \frac{d\theta \cos.n\theta}{\cos.\theta^3} \quad \text{et} \quad y = a\int \frac{d\theta \sin.n\theta}{\cos.\theta^3}.$$

15. Iam per reductionem formularum integralium, quali supra sum usus, reperiemus

$$\int \frac{d\theta \cos.n\theta}{\cos.\theta^3} = \frac{2\sin.(n-1)\theta}{(n-3)\cos.\theta^2} - \frac{n+1}{n-3}\int \frac{d\theta \cos.(n-2)\theta}{\cos.\theta^3},$$

$$\int \frac{d\theta \sin.n\theta}{\cos.\theta^3} = \frac{-2\cos.(n-1)\theta}{(n-3)\cos.\theta^2} - \frac{n+1}{n-3}\int \frac{d\theta \sin.(n-2)\theta}{\cos.\theta^3},$$

unde patet, si integratio succedat casu quocunque $n=\lambda$, eam quoque succedere casibus $n=\lambda+2$, $n=\lambda+4$, $n=\lambda+6$ etc. sicque infinitas curvas algebraicas ex unica impetrari. Patet autem, si sit $n=1$, fore

$$\int \frac{d\theta \cos.\theta}{\cos.\theta^3} = \frac{\sin.2\theta}{2\cos.\theta^2} \quad \text{et} \quad \int \frac{d\theta \sin.\theta}{\cos.\theta^3} = -\frac{\cos.2\theta}{2\cos.\theta^2}$$

sive

$$\int \frac{d\theta \cos.\theta}{\cos.\theta^3} = \frac{\sin.\theta}{\cos.\theta} \quad \text{et} \quad \int \frac{d\theta \sin.\theta}{\cos.\theta^3} = +\frac{1}{2\cos.\theta^2},$$

quo casu prodit

$$x = \frac{a\sin.\theta}{\cos.\theta} \quad \text{et} \quad y = \frac{a}{2\cos.\theta^2},$$

ergo $\frac{xx}{2a} = \frac{a\sin.\theta^2}{2\cos.\theta^2}$ hincque

$$y - \frac{xx}{2a} = \frac{a}{2},$$

quae est aequatio pro ipsa parabola.

16. Verum etiamsi hic unum casum integrabilitatis, quo $\varphi = \theta$ seu $n = 1$, habeamus cognitum, tamen singulari fato ex eo nulli alii casus elici possunt. Si enim statuamus $n = 3$, ob denominatorem $n - 3$ evanescentem integralia inde pro casu $\varphi = 3\theta$ minime reperiuntur. Casu autem $n = -1$ formulae praecedentes redeunt, ita ut propter hoc incommodum nullus aditus ad curvas magis compositas pateat. Videri ergo posset parabola pari conditione praedita ac circulus, ut praeter se ipsam nullas alias agnoscat curvas algebraicas secum commensurabiles. Ex ipsa verum angulorum compositione manifestum est, quicunque numerus integer excepta unitate pro n statuatur, formulam $\int \frac{d\theta \cos. n\theta}{\cos. \theta^3}$ nunquam integrabilem evadere, sed semper per integrationem ipsum angulum θ induci. Interim tamen alia methodo quaesito satisfieri potest, unde non difficulter talis curva eruitur

$$x = \tfrac{1}{2} z \sqrt{(4 + zz)} \quad \text{et} \quad y = \sqrt{(4 + zz)}$$

seu

$$y^4 = 4(xx + yy),$$

pro qua est

$$\sqrt{(dx^2 + dy^2)} = dz\sqrt{(1 + zz)}.$$

DE ELLIPSI

17. Progredior ergo ad curvas algebraicas indagandas, quarum arcus cum arcubus ellipseos sint commensurabiles. Quaestio igitur huc redit, ut curvarum inveniendarum arcus exprimantur per hanc formulam $\int dz \sqrt{\left(1 + \frac{mmzz}{1 - zz}\right)}$, quae est formula pro arcu elliptico abscissae z respondente, dum applicata est $= m\sqrt{(1 - zz)}$. Pro curvis ergo, quas quaerimus, statuamus coordinatas

$$x = \int dz \cos. \varphi \sqrt{\left(1 + \frac{mmzz}{1 - zz}\right)} \quad \text{et} \quad y = \int dz \sin. \varphi \sqrt{\left(1 + \frac{mmzz}{1 - zz}\right)}$$

et videamus, quomodo angulus φ capi debeat, ut ambae istae formulae fiant integrabiles. Ponamus

$$z = \sin. \theta$$

et hae formulae erunt

$$x = \int d\theta \cos. \varphi \sqrt{(\cos. \theta^2 + mm \sin. \theta^2)} \quad \text{et} \quad y = \int d\theta \sin. \varphi \sqrt{(\cos. \theta^2 + mm \sin. \theta^2)},$$

ubi manifestum est, quaecunque multipla anguli θ pro φ statuantur, has ex-

pressiones nullo modo ad integrationem perduci posse. Aliis ergo artificiis erit utendum, siquidem certum est dari curvas algebraicas quaesito satisfacientes.

18. Quoniam irrationalitas negotium turbat, ad eius speciem saltem tollendam pono

$$m \operatorname{tang.} \theta = \operatorname{tang.} \omega,$$

ut sit

$$mm \sin. \theta^2 = \cos. \theta^2 \operatorname{tang.} \omega^2$$

hincque

$$\sqrt{(\cos. \theta^2 + mm \sin. \theta^2)} = \cos. \theta \sqrt{(1 + \operatorname{tang.} \omega^2)} = \frac{\cos. \theta}{\cos. \omega}.$$

Hac substitutione facta nostrae coordinatae erunt

$$x = \int \frac{d\theta \cos. \theta \cos. \varphi}{\cos. \omega} \quad \text{et} \quad y = \int \frac{d\theta \cos. \theta \sin. \varphi}{\cos. \omega},$$

ubi notandum est angulos θ et ω ita a se invicem pendere, ut sit

$$m \operatorname{tang.} \theta = \operatorname{tang.} \omega \quad \text{ideoque} \quad \frac{m\,d\theta}{\cos. \theta^2} = \frac{d\omega}{\cos. \omega^2}.$$

Statuatur iam

$$\varphi = n\theta - \omega$$

et ob

$$\cos. \varphi = \cos. n\theta \cos. \omega + \sin. n\theta \sin. \omega$$

et

$$\sin. \varphi = \sin. n\theta \cos. \omega - \cos. n\theta \sin. \omega$$

coordinatae ita exprimentur, ut sit ob $\operatorname{tang.} \omega = m \operatorname{tang.} \theta$

$$\begin{aligned} x &= \int d\theta \cos. \theta \cos. n\theta + \int d\theta \cos. \theta \sin. n\theta \operatorname{tang.} \omega \\ &= \int d\theta (\cos. \theta \cos. n\theta + m \sin. \theta \sin. n\theta), \\ y &= \int d\theta \cos. \theta \sin. n\theta - \int d\theta \cos. \theta \cos. n\theta \operatorname{tang.} \omega \\ &= \int d\theta (\cos. \theta \sin. n\theta - m \sin. \theta \cos. n\theta), \end{aligned}$$

quas formulas, quicunque numerus pro n assumatur praeter unitatem, manifestum est semper esse integrabiles.

19. Cum igitur sit

$$\cos.\theta\cos.n\theta = \frac{1}{2}\cos.(n-1)\theta + \frac{1}{2}\cos.(n+1)\theta,$$

$$\sin.\theta\sin.n\theta = \frac{1}{2}\cos.(n-1)\theta - \frac{1}{2}\cos.(n+1)\theta,$$

$$\cos.\theta\sin.n\theta = \frac{1}{2}\sin.(n-1)\theta + \frac{1}{2}\sin.(n+1)\theta,$$

$$-\sin.\theta\cos.n\theta = \frac{1}{2}\sin.(n-1)\theta - \frac{1}{2}\sin.(n+1)\theta,$$

substituendis his valoribus habebimus

$$x = \frac{1}{2}\int d\theta\,((m+1)\cos.(n-1)\theta - (m-1)\cos.(n+1)\theta),$$

$$y = \frac{1}{2}\int d\theta\,((m+1)\sin.(n-1)\theta - (m-1)\sin.(n+1)\theta),$$

unde valores integrales sponte fluunt

$$x = +\frac{(m+1)\sin.(n-1)\theta}{2(n-1)} - \frac{(m-1)\sin.(n+1)\theta}{2(n+1)},$$

$$y = -\frac{(m+1)\cos.(n-1)\theta}{2(n-1)} + \frac{(m-1)\cos.(n+1)\theta}{2(n+1)}.$$

Hincque, cum pro n numeros quoscunque rationales praeter unitatem accipere liceat, innumerabiles lineae algebraicae exhiberi possunt.

20. Cum igitur unitas pro n substitui nequeat, casus simplicissimus prodibit, si ponatur $n = 0$, quo ergo habebitur

$$x = \frac{1}{2}(m+1)\sin.\theta - \frac{1}{2}(m-1)\sin.\theta = \sin.\theta,$$

$$y = \frac{1}{2}(m+1)\cos.\theta + \frac{1}{2}(m-1)\cos.\theta = m\cos.\theta,$$

unde fit

$$mmxx + yy = mm \quad \text{ideoque} \quad y = m\sqrt{(1-xx)},$$

quae est aequatio pro ellipsi proposita, cuius arcus ob $x = \sin.\theta = z$ utique est

$$\int dz\sqrt{\left(1 + \frac{mmzz}{1-zz}\right)},$$

uti requiritur; erit enim $x = z$ et $y = m\sqrt{(1 - zz)}$. Aliae vero curvae, quarum eadem est rectificatio, prodibunt, si numero n praeter unitatem alii valores tribuantur. Sit igitur $n = 2$ atque habebitur

$$x = +\frac{1}{2}(m+1)\sin.\theta - \frac{1}{6}(m-1)\sin.3\theta,$$

$$y = -\frac{1}{2}(m+1)\cos.\theta + \frac{1}{6}(m-1)\cos.3\theta,$$

unde fit

$$xx + yy = \frac{1}{4}(m+1)^2 + \frac{1}{36}(m-1)^2 - \frac{1}{6}(mm-1)\cos.2\theta$$

seu

$$xx + yy = \frac{5}{18}mm + \frac{4}{9}m + \frac{5}{18} - \frac{1}{6}(mm-1)\cos.2\theta.$$

Verum praestat uti formulis illis pro x et y inventis, quia ad cognoscendam et construendam curvam sunt maxime idoneae.

21. Antequam in evolutione horum casuum ulterius progrediar, notari conveniet quantitatem m tam negative quam affirmative capi posse, propterea quod in expressione arcus quadratum mm tantum inest. Veruntamen iidem casus resultant, si numerus n negative capiatur, ita ut quantitate m ambigua assumta non opus sit pro n valores negativos statuere. Hinc ergo quilibet numerus positivus pro n sumtus duas praebet lineas algebraicas, prouti m vel affirmative accipitur vel negative; sicque post ellipsin has duas habebimus curvas satisfacientes

$$x = \frac{1}{2}(m+1)\sin.\theta - \frac{1}{6}(m-1)\sin.3\theta,$$

$$y = \frac{1}{2}(m+1)\cos.\theta - \frac{1}{6}(m-1)\cos.3\theta,$$

$$x = \frac{1}{2}(m-1)\sin.\theta - \frac{1}{6}(m+1)\sin.3\theta,$$

$$y = \frac{1}{2}(m-1)\cos.\theta - \frac{1}{6}(m+1)\cos.3\theta,$$

ubi quidem valorem ipsius y negative sumsi. Similes fere expressiones prodeunt, si ponatur $n = \frac{1}{2}$, unde quoque hae duae curvae oriuntur

$$x = (m+1)\sin.\frac{1}{2}\theta - \frac{1}{3}(m-1)\sin.\frac{3}{2}\theta,$$
$$y = (m+1)\cos.\frac{1}{2}\theta + \frac{1}{3}(m-1)\cos.\frac{3}{2}\theta,$$

$$x = (m-1)\sin.\frac{1}{2}\theta - \frac{1}{3}(m+1)\sin.\frac{3}{2}\theta,$$
$$y = (m-1)\cos.\frac{1}{2}\theta + \frac{1}{3}(m+1)\cos.\frac{3}{2}\theta.$$

Atque evidens est eliminando arcu θ has quatuor aequationes ad eundem ordinem esse ascensuras.

22. Ponamus $n = 3$ hincque duae nascentur curvae istae

$$x = \frac{1}{4}(m+1)\sin.2\theta - \frac{1}{8}(m-1)\sin.4\theta,$$
$$y = \frac{1}{4}(m+1)\cos.2\theta - \frac{1}{8}(m-1)\cos.4\theta,$$

$$x = \frac{1}{4}(m-1)\sin.2\theta - \frac{1}{8}(m+1)\sin.4\theta,$$
$$y = \frac{1}{4}(m-1)\cos.2\theta - \frac{1}{8}(m+1)\cos.4\theta.$$

At si ponamus $n = \frac{1}{3}$, non multum absimiles hae curvae nascuntur

$$x = \frac{3}{4}(m+1)\sin.\frac{2}{3}\theta - \frac{3}{8}(m-1)\sin.\frac{4}{3}\theta,$$
$$y = \frac{3}{4}(m+1)\cos.\frac{2}{3}\theta + \frac{3}{8}(m-1)\cos.\frac{4}{3}\theta,$$

$$x = \frac{3}{4}(m-1)\sin.\frac{2}{3}\theta - \frac{3}{8}(m+1)\sin.\frac{4}{3}\theta,$$
$$y = \frac{3}{4}(m-1)\cos.\frac{2}{3}\theta + \frac{3}{8}(m+1)\cos.\frac{4}{3}\theta.$$

Omnes enim hae quatuor curvae tantum ad ordinem linearum quartum referuntur. Ex quibus perspicuum est, quomodo ex quavis hypothesi quaternae curvae elici queant ad eundem ordinem referendae, nisi quatenus forte casu ordo deprimi possit. Haec ergo infinita linearum algebraicarum

multitudo, quarum arcus omnes per arcus ellipticos absolute mensurantur, omnino est notatu digna idque eo magis, quod pro omnibus coordinatae x et y binis tantum terminis exprimuntur; unde earum constructio haud parum concinna adornari potest, etiamsi plerumque curvae ad altiores linearum ordines referantur.

23. De his autem omnibus lineis imprimis est notandum eas ad classem epicycloidum et hypocycloidum pertinere ac per motum volutorium circuli super peripheria alterius circuli sive extus sive intus describi posse. Hoc autem hae curvae a vulgaribus epicycloidibus et hypocycloidibus differunt, quod in circulo mobili punctum describens non in eius peripheria, sed sive extra sive intra eam assumi debet. Si enim in peripheria caperetur, quo casu epicycloides et hypocycloides vulgares prodirent, curvae descriptae absolute essent rectificabiles neque idcirco ad nostrum institutum essent accommodatae; sin autem punctum describens in ipso centro circuli mobilis assumeretur curva descripta perpetuo foret circulus. Verum sive punctum describens capiatur extra sive intra peripheriam circuli mobilis, hoc modo semper curvae describuntur, quarum rectificatio per arcus ellipticos absolute confici potest. Nostrae ergo curvae prodibunt, si distantia puncti describentis a centro circuli mobilis sive maior fuerit sive minor quam eius semidiameter.

24. Natura autem huiusmodi linearum accuratius perpensa curvae, quarum arcus per arcus datae ellipsis mensurantur, ita describi posse deprehendentur. Sit in ellipsi proposita ratio amborum axium principalium $= 1 : m$ ac posito radio circuli mobilis $= r$ capiatur distantia puncti describentis ab eius centro sive $= \frac{m+1}{m-1}r$ sive $= \frac{m-1}{m+1}r$. Tum si iste circulus super quocunque alio circulo sive extus sive intus provolvatur, ab utroque puncto describente semper eiusmodi curva describetur, cuius rectificatio cum rectificatione ellipsis propositae conveniet. Quo autem curvae hoc modo descriptae fiant algebraicae, necesse est, ut radius circuli mobilis ad radium circuli immoti rationem teneat rationalem, quae quo fuerit simplicior, eo minus curvae descriptae erunt compositae; ac constituto quidem circulo immoto, sive mobilis extra eum sive intra volvatur, tum vero, sive punctum describens extra sive intra circulum mobilem accipiatur, quaternae illae curvae describentur, quas coniunctas inveneramus.

25. Operae pretium fore videtur harum linearum epi- et hypocycloidalium proprietates primarias, quatenus huc pertinent, ac praecipue earum rectificationem attentius contemplari. Sit igitur C (Fig. 1, 2, item 3, 4)

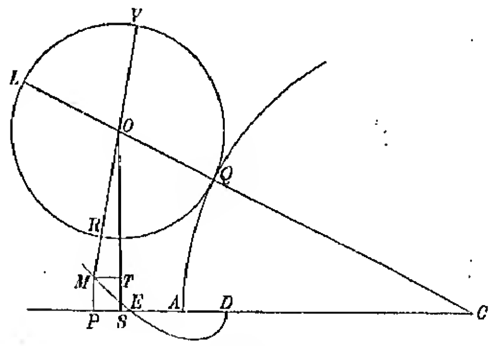

Fig. 1.

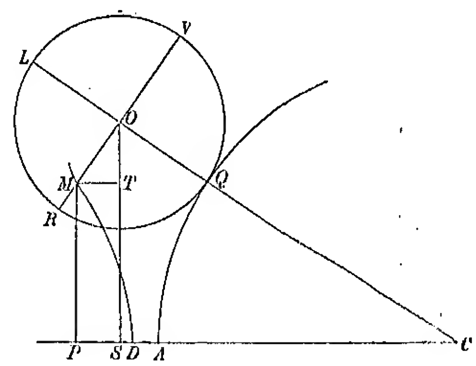

Fig. 2.

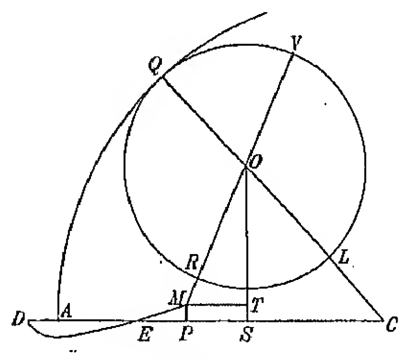

Fig. 3.

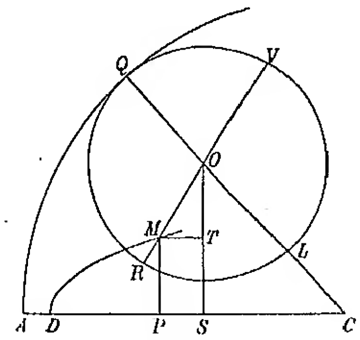

Fig. 4.

centrum circuli immoti AQ eiusque radius $CA = CQ = a$, super cuius peripheria volvatur circulus $OLRQV$, cuius radius $OQ = OR = r$; sitque punctum describens M in radio OR ac vocetur $OM = \mu r$, ita ut sit sive $\mu = \frac{m+1}{m-1}$ sive $\mu = \frac{m-1}{m+1}$. Hoc modo a stilo M descripta sit curva DM, cuius initium D ei respondeat circuli mobilis situi, quo punctum R tangebat circulum immotum in A. Hinc ergo ex natura motus volutorii erit arcus QR aequalis arcui QA. Quare si dicamus angulum $ACQ = \varphi$, ob arcum $AQ = QR = a\varphi$ erit angulus $QOR = \frac{a}{r}\varphi$. Vocemus autem brevitatis gratia hunc angulum $QOR = \alpha\varphi$, ut sit $\alpha = \frac{a}{r}$. Tum vero ex punctis M et O ad rectam CA pro axe assumtam demittantur perpendicula MP et OS itemque ex M in rectam MT axi AC parallelam sintque coordinatae orthogonales curvae descriptae $CP = x$ et $PM = y$.

26. Cum iam sit angulus $ACQ = \varphi$ et $CO = a \pm r$, ubi signum superius pro curvis epicycloidalibus, inferius vero pro hypocycloidalibus valet, erit $CS = (a \pm r)\cos.\varphi$ et $OS = (a \pm r)\sin.\varphi$. Deinde ob ang. $COS = 90^\circ - \varphi$ et $OR = \alpha\varphi$ erit ang. $MOT = (\alpha + 1)\varphi - 90^\circ$ pro epicycloidalibus (Fig. 1 et 2) ; pro hypocycloidalibus (Fig. 3 et 4) ob $COS = 90^\circ - \varphi$ et $COR = 180^\circ - \alpha\varphi$ it ang. $MOT = 90^\circ - (\alpha - 1)\varphi$, unde ex triangulo OMT ad T rectangulo b latus $OM = \mu r$ obtinebimus pro utroque casu curvarum epicycloidalium Fig. 1 et 2)

$$MT = -\mu r\cos.(\alpha + 1)\varphi,$$

$$OT = +\mu r\sin.(\alpha + 1)\varphi,$$

rgo

$$CP = (a + r)\cos.\varphi - \mu r\cos.(\alpha + 1)\varphi = x,$$

$$PM = (a + r)\sin.\varphi - \mu r\sin.(\alpha + 1)\varphi = y$$

t curvarum hypocycloidalium (Fig. 3 et 4)

$$MT = \mu r\cos.(\alpha - 1)\varphi,$$

$$OT = \mu r\sin.(\alpha - 1)\varphi,$$

rgo

$$CP = (a - r)\cos.\varphi + \mu r\cos.(\alpha - 1)\varphi = x,$$

$$PM = (a - r)\sin.\varphi - \mu r\sin.(\alpha - 1)\varphi = y.$$

Consequenter pro utroque casu coniunctim

$$CP = x = (a \pm r)\cos.\varphi \mp \mu r\cos.\left(1 \pm \frac{a}{r}\right)\varphi,$$

$$PM = y = (a \pm r)\sin.\varphi \mp \mu r\sin.\left(1 \pm \frac{a}{r}\right)\varphi.$$

27. Hinc ergo videmus totum discrimen inter has curvas epicycloidales et hypocycloidales tantum in signo quantitatis r esse situm, ita ut omnes his expressionibus pro coordinatis $CP = x$ et $PM = y$ possimus complecti

$$x = (a + r)\cos.\varphi - \mu r\cos.\left(1 + \frac{a}{r}\right)\varphi,$$

$$y = (a + r)\sin.\varphi - \mu r\sin.\left(1 + \frac{a}{r}\right)\varphi,$$

quae proprie ad epicycloidales pertinent, sed sumta quantitate r negativa simul ad hypocycloidales extenduntur. Differentiando ergo habebimus

$$dx = -(a+r)d\varphi\left(\sin.\varphi - \mu\sin.\left(1+\frac{a}{r}\right)\varphi\right),$$

$$dy = +(a+r)d\varphi\left(\cos.\varphi - \mu\cos.\left(1+\frac{a}{r}\right)\varphi\right),$$

unde elementum arcus huius curvae $\sqrt{(dx^2+dy^2)} = ds$ reperitur

$$ds = (a+r)d\varphi\sqrt{\left(1+\mu\mu-2\mu\cos.\frac{a}{r}\varphi\right)},$$

et radius osculi in M ita erit expressus

$$\frac{(a+r)\left(1+\mu\mu-2\mu\cos.\frac{a}{r}\varphi\right)^{\frac{3}{2}}}{1+\mu\mu-\mu\left(2+\frac{a}{r}\right)\cos.\frac{a}{r}\varphi}.$$

28. Quacunque igitur huiusmodi curva descripta dabitur ellipsis, in qua arcui curvae DM arcus aequalis assignari poterit. Sit $adbe$ (Fig. 5) haec ellipsis eiusque axes orthogonales ab et de; vocetur semiaxis minor $ca = cb = c$ et semiaxis maior $cd = ce = mc$ sumtaque super illo a centro c abscissa $cp = z$ erit applicata $pm = m\sqrt{(cc-zz)}$ et arcus ellipticus

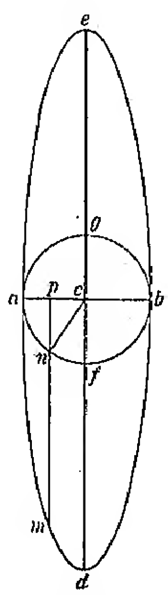

Fig. 5.

$$dm = \int dz\sqrt{\left(1+\frac{mmzz}{cc-zz}\right)}.$$

Statuatur $z = c\sin.\theta$ eritque hic arcus

$$dm = \int cd\theta\sqrt{(1+(mm-1)\sin.\theta^2)}$$

$$= \int cd\theta\sqrt{\left(\frac{1}{2}(mm+1)-\frac{1}{2}(mm-1)\cos.2\theta\right)};$$

quae forma ut illi pro ds inventae aequalis reddatur, fieri oportet

$$\theta = \frac{a}{2r}\varphi = \frac{1}{2}QOR \quad \text{et} \quad \frac{mm+1}{mm-1} = \frac{1+\mu\mu}{2\mu} \quad \text{seu} \quad m = \frac{\mu+1}{\mu-1}$$

vel, quod eodem redit, capiatur $m = \frac{VM}{RM}$ in fig. 1, 2, 3, 4 eritque arcus ellipticus

$$dm = \int \frac{acd\varphi}{2(\mu-1)r} \sqrt{\left(1 + \mu\mu - 2\mu \cos. \frac{a}{r}\varphi\right)}.$$

Superest ergo, ut sit

$$\frac{ac}{2(\mu-1)r} = a + r,$$

unde semiaxes ellipsis fiunt

$$ca = cb = \frac{2(\mu-1)r(a+r)}{a} \quad \text{et} \quad cd = ce = \frac{2(\mu+1)r(a+r)}{a}.$$

29. In genere ergo habebimus hanc constructionem pro ellipsi quaesita

$$\text{semiaxis} \quad ca = cb = \frac{2RM \cdot CO}{CQ} \quad \text{et semiaxis} \quad cd = ce = \frac{2VM \cdot CO}{CQ},$$

qua descripta circa centrum C radio $ca = cb$ delineetur circulus $afbg$, tum ducatur radius cn ita, ut sit angulus $fcn = \frac{1}{2} QOR$, et per n ducta recta pnm axi maiori de parallela erit arcus ellipticus dm aequalis arcui curvae supra descriptae DM. Unde patet, si circulus mobilis iam per semiperipheriam fuerit provolutus, quod evenit, cum punctum V circulo immoto applicabitur, tum longitudinem curvae descriptae aequalem fore quadranti elliptico dma. Cum autem circulus mobilis integram revolutionem absolverit, tractus curvae descriptae semiperipheriae ellipticae dae erit aequalis; sicque uti ellipsis est curva in se rediens, ita provolutione continuata longitudo curvae continuo crescet.

30. De his curvis adhuc notari meretur ipsam quoque ellipsin inter eas comprehendi. Si enim pro hypocycloidalibus sumatur radius circuli immoti aequalis diametro circuli mobilis seu $a = 2r$ vel si in nostris formulis § 27 ponamus $r = -\frac{1}{2}a$, habebimus

$$x = \frac{1}{2}a \cos. \varphi + \frac{1}{2}\mu a \cos. \varphi = -(1 + \mu) r \cos. \varphi,$$

$$y = \frac{1}{2}a \sin. \varphi - \frac{1}{2}\mu a \sin. \varphi = -(1 - \mu) r \sin. \varphi,$$

unde prodit

$$\frac{xx}{(1+\mu)^2}+\frac{yy}{(1-\mu)^2}=rr,$$

quae est aequatio pro ellipsi, cuius semiaxes sunt $(\mu-1)r$ et $(\mu+1)r$ seu MR et MV, estque ea ipsa ellipsis, cuius arcubus nostrae curvae mensurantur; nam ob $CQ=2CO$ fit utique $ca=RM$ et $cd=VM$. Potest itaque quaecunque ellipsis provolutione circuli intra peripheriam alterius circuli, cuius radius duplo est maior, describi; ubicunque enim tum stylus in circulo mobili figatur, ab eo ellipsis describetur.

31. Innumerabiles autem curvae, quae sint cum arcubus parabolicis commensurabiles, quarum supra [§ 16] unam exhibui, seu ut positis coordinatis x et y sit

$$\sqrt{(dx^2+dy^2)}=dz\sqrt{(1+zz)},$$

sequenti modo se habebunt. Ponatur

$$z=\frac{2}{n}\operatorname{tang.}\varphi \quad \text{seu} \quad \operatorname{tang.}\varphi=\frac{1}{2}nz$$

ac statuatur

$$x=\frac{2\sin.n\varphi}{nn\cos.\varphi^2} \quad \text{et} \quad y=\frac{2\cos.n\varphi}{nn\cos.\varphi^2};$$

erit semper, quicunque numerus pro n assumatur,

$$\int\sqrt{(dx^2+dy^2)}=\int dz\sqrt{(1+zz)}.$$

Facile autem angulus φ eliminatur ob $\sqrt{(xx+yy)}=\frac{2}{nn\cos.\varphi^2}$, unde fit

$$\cos.\varphi=\frac{\sqrt{2}}{n\sqrt[4]{(xx+yy)}}$$

hincque

$$\frac{y}{\sqrt{(xx+yy)}}=\cos.n\varphi.$$

At si variabilem z retinere velimus, erit

$$x=\frac{\frac{n}{1}\cdot\frac{nz}{2}-\frac{n(n-1)(n-2)}{1\cdot2\cdot3}\cdot\frac{n^3z^3}{8}+\frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot2\cdot3\cdot4\cdot5}\cdot\frac{n^5z^5}{32}-\text{etc.}}{\frac{1}{2}nn\left(1+\frac{nnzz}{4}\right)^{\frac{n-2}{2}}},$$

$$y = \frac{1 - \frac{n(n-1)}{1\cdot 2}\cdot\frac{nnzz}{4} + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{n^4z^4}{16} - \text{etc.}}{\frac{1}{2}nn\left(1+\frac{nnzz}{4}\right)^{\frac{n-2}{2}}},$$

quae formulae, quoties n sumitur numerus integer positivus, finito terminorum numero constabunt. Verum priores semper, etiamsi pro n statuatur numerus fractus, ad aequationem finitam deducunt. Veluti si $n = \frac{1}{2}$, cum sit

$$\cos.\varphi = \frac{2\sqrt{2}}{\sqrt[4]{(xx+yy)}} \quad \text{et} \quad \cos.\frac{1}{2}\varphi = \frac{y}{\sqrt{(xx+yy)}},$$

erit hinc

$$\cos.\varphi = \frac{2yy}{xx+yy} - 1 = \frac{yy-xx}{xx+yy},$$

unde obtinetur

$$\frac{64}{xx+yy} = \frac{(yy-xx)^4}{(xx+yy)^4}$$

seu

$$(yy-xx)^4 = 64\,(xx+yy)^3$$

pro linea ordinis octavi.

DE COMPARATIONE ARCUUM CURVARUM IRRECTIFICABILIUM

Commentatio 818 indicis ENESTROEMIANI
Opera postuma 1, Petropoli 1862, p. 452—486

SECTIO PRIMA

CONTINENS EVOLUTIONEM HUIUS AEQUATIONIS

$$0 = \alpha + 2\beta(x + y) + \gamma(xx + yy) + 2\delta xy$$

I

Si ex hac aequatione singillatim utriusque variabilis x et y valor exti hatur, reperietur

$$y = \frac{-\beta - \delta x + \sqrt{(\beta\beta - \alpha\gamma + 2\beta(\delta - \gamma)x + (\delta\delta - \gamma\gamma)xx)}}{\gamma},$$

$$x = \frac{-\beta - \delta y - \sqrt{(\beta\beta - \alpha\gamma + 2\beta(\delta - \gamma)y + (\delta\delta - \gamma\gamma)yy)}}{\gamma}.$$

Ponatur brevitatis gratia

$$\beta\beta - \alpha\gamma = Ap, \quad \beta(\delta - \gamma) = Bp \quad \text{et} \quad \delta\delta - \gamma\gamma = Cp$$

eritque

$$\beta + \gamma y + \delta x = +\sqrt{p(A + 2Bx + Cxx)},$$

$$\beta + \gamma x + \delta y = -\sqrt{p(A + 2By + Cyy)}.$$

II

Litteris iam A, B, C pro lubitu assumtis ex iis litterae α, β, γ, δ et p sequenti modo definientur. Primo ex aequalitate secunda fit $\delta - \gamma = \frac{Bp}{\beta}$,

qui valor in tertia $\delta+\gamma=\frac{Cp}{\delta-\gamma}$ substitutus dat $\delta+\gamma=\frac{C\beta}{B}$, ita ut sit

$$\delta=\frac{C\beta}{2B}+\frac{Bp}{2\beta} \quad \text{et} \quad \gamma=\frac{C\beta}{2B}-\frac{Bp}{2\beta}.$$

Hinc autem aequalitas prima abit in hanc

$$\beta\beta-\frac{C\alpha\beta}{2B}+\frac{B\alpha p}{2\beta}=Ap,$$

ex qua definietur

$$p=\frac{\beta\beta(2B\beta-C\alpha)}{B(2A\beta-B\alpha)}$$

indeque porro

$$\delta=\frac{\beta(AC\beta+BB\beta-BC\alpha)}{B(2A\beta-B\alpha)} \quad \text{et} \quad \gamma=\frac{\beta\beta(AC-BB)}{B(2A\beta-B\alpha)}.$$

Sic ergo litterae α et β arbitrio nostro relinquuntur, quarum altera quidem unitate exprimi poterit, altera vero constantem arbitrariam a coefficientibus A, B, C non pendentem exhibebit.

III

Differentietur nunc aequatio proposita ac prodibit

$$dx(\beta+\gamma x+\delta y)+dy(\beta+\gamma y+\delta x)=0,$$

unde conficitur haec aequatio

$$\frac{dx}{\beta+\gamma y+\delta x}=\frac{-dy}{\beta+\gamma x+\delta y},$$

quae substitutis valoribus in articulo I inventis abibit in hanc aequationem differentialem

$$\frac{dx}{\sqrt{(A+2Bx+Cxx)}}-\frac{dy}{\sqrt{(A+2By+Cyy)}}=0,$$

cuius propterea integralis est ipsa aequatio assumta.

IV

Proposita ergo vicissim hac aequatione differentiali

$$\frac{dx}{\sqrt{(A+2Bx+Cxx)}}-\frac{dy}{\sqrt{(A+2By+Cyy)}}=0$$

eius integrale semper algebraice exhiberi poterit, quippe quod erit

$$0 = \alpha + 2\beta(x+y) + \frac{\beta\beta(AC-BB)(xx+yy) + 2\beta(AC\beta + BB\beta - BC\alpha)xy}{B(2A\beta - B\alpha)},$$

et quia hic continetur constans ab arbitrio nostro pendens, erit hoc integrale quoque completum aequationis differentialis propositae. Erit ergo retentis litteris graecis

$$\text{vel } y = \frac{-\beta - \delta x + \sqrt{p(A+2Bx+Cxx)}}{\gamma}$$

$$\text{vel } x = \frac{-\beta - \delta y - \sqrt{p(A+2By+Cyy)}}{\gamma}.$$

V

Quemadmodum autem istarum formularum integralium differentia

$$\int \frac{dx}{\sqrt{(A+2Bx+Cxx)}} - \int \frac{dy}{\sqrt{(A+2By+Cyy)}}$$

est constans, siquidem inter x et y ea relatio subsistat, ut sit

$$0 = \alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy,$$

ita etiam eadem manente relatione differentia huiusmodi formularum

$$\int \frac{x^n dx}{\sqrt{(A+2Bx+Cxx)}} - \int \frac{y^n dy}{\sqrt{(A+2By+Cyy)}}$$

commode exprimi potest; quos valores indagasse operae pretium erit.

VI

Posito ergo exponente $n = 1$ statuamus

$$\frac{xdx}{\sqrt{(A+2Bx+Cxx)}} - \frac{ydy}{\sqrt{(A+2By+Cyy)}} = dV$$

eritque valoribus initio traditis pro his formulis irrationalibus substituendis

$$\frac{xdx\sqrt{p}}{\beta+\gamma y+\delta x} + \frac{ydy\sqrt{p}}{\beta+\gamma x+\delta y} = dV$$

seu

$$xdx(\beta+\gamma x+\delta y)+ydy(\beta+\gamma y+\delta x)=\frac{dV}{\sqrt{p}}(\beta+\gamma y+\delta x)(\beta+\gamma x+\delta y);$$

at est

$$(\beta+\gamma y+\delta x)(\beta+\gamma x+\delta y)=\beta\beta+\beta(\gamma+\delta)(x+y)+\gamma\delta(xx+yy)+(\gamma\gamma+\delta\delta)xy.$$

VII

Quo hanc formulam facilius expediamus, ponamus

$$x+y=t \quad \text{et} \quad xy=u;$$

erit

$$xx+yy=tt-2u \quad \text{et} \quad x^3+y^3=t^3-3tu$$

sicque aequatio abit in hanc formam

$$\beta(xdx+ydy)+\gamma(xxdx+yydy)+\delta xy(dx+dy)$$
$$=\frac{dV}{\sqrt{p}}(\beta\beta+\beta(\gamma+\delta)t+\gamma\delta tt+(\gamma-\delta)^2u).$$

Ipsa autem aequatio assumta fit

$$0=\alpha+2\beta t+\gamma tt+2(\delta-\gamma)u$$

et penitus introductis litteris t et u habebimus

$$\beta(tdt-du)+\gamma(ttdt-tdu-udt)+\delta udt=\frac{dV}{\sqrt{p}}(\beta\beta-\alpha\delta+\beta(\gamma-\delta)t+(\gamma\gamma-\delta\delta)u)$$

seu

$$dt(\beta t+\gamma tt-(\gamma-\delta)u)-du(\beta+\gamma t)=\frac{dV}{\sqrt{p}}(\beta\beta-\alpha\delta+\beta(\gamma-\delta)t+(\gamma\gamma-\delta\delta)u).$$

VIII

Ex aequatione autem assumta, si differentietur, fit

$$dt(\beta+\gamma t)=(\gamma-\delta)du,$$

unde aequationis ultimae prius membrum transformatur in

$$\frac{dt}{\gamma-\delta}(-\beta\beta-\beta(\gamma+\delta)t-\gamma\delta tt-(\gamma-\delta)^2u);$$

quod cum aequale esse debeat huic formulae

$$\frac{dV}{\sqrt{p}}(\beta\beta+\beta(\gamma+\delta)t+\gamma\delta tt+(\gamma-\delta)^2u),$$

commode inde oritur

$$\frac{dV}{\sqrt{p}}=\frac{-dt}{\gamma-\delta}\quad\text{et}\quad V=\frac{-t\sqrt{p}}{\gamma-\delta}.$$

IX

Cum iam sit $t=x+y$, habebimus sequentem aequationem integratam

$$\int\frac{xdx}{\sqrt{(A+2Bx+Cxx)}}-\int\frac{ydy}{\sqrt{(A+2By+Cyy)}}=\text{Const.}-\frac{(x+y)\sqrt{p}}{\gamma-\delta}$$

existente

$$0=\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy,$$

siquidem relationes supra exhibitae inter litteras A, B, C et α, β, γ, δ ac p locum habeant. Hinc ergo eadem manente determinatione variabilium x et y erit generalius

$$\int\frac{dx(\mathfrak{A}+\mathfrak{B}x)}{\sqrt{(A+2Bx+Cxx)}}-\int\frac{dy(\mathfrak{A}+\mathfrak{B}y)}{\sqrt{(A+2By+Cyy)}}=\text{Const.}-\frac{\mathfrak{B}(x+y)\sqrt{p}}{\gamma-\delta}.$$

X

Progrediamur porro ac statuamus

$$\frac{xxdx}{\sqrt{(A+2Bx+Cxx)}}-\frac{yydy}{\sqrt{(A+2By+Cyy)}}=dV;$$

erit posito brevitatis ergo

$$\beta\beta+\beta(\gamma+\delta)t+\gamma\delta tt+(\gamma-\delta)^2u=T,$$

si loco istarum formularum surdarum valores ante reperti substituantur,

$$xxdx(\beta+\gamma x+\delta y)+yydy(\beta+\gamma y+\delta x)=\frac{TdV}{\sqrt{p}}$$

existente ut ante $t=x+y$ et $u=xy$.

XI

Cum nunc sit $x^4+y^4=t^4-4ttu+2uu$, erit eliminatis variabilibus x et y

$$\beta(ttdt-tdu-udt)+\gamma(t^3dt-ttdu-2tudt+udu)+\delta u(tdt-du)=\frac{TdV}{\sqrt{p}}$$

sive

$$dt(\beta tt-\beta u+\gamma t^3-2\gamma tu+\delta tu)-du(\beta t+\gamma tt-\gamma u+\delta u)=\frac{TdV}{\sqrt{p}}.$$

Cum autem sit

$$du=\frac{dt(\beta+\gamma t)}{\gamma-\delta},$$

erit hac facta substitutione

$$\frac{dt}{\gamma-\delta}(-\beta\beta t-\beta(\gamma+\delta)tt-\gamma\delta t^3-(\gamma-\delta)^2tu)=\frac{TdV}{\sqrt{p}}=\frac{-Ttdt}{\gamma-\delta}$$

sicque erit

$$\frac{dV}{\sqrt{p}}=\frac{-tdt}{\gamma-\delta}\quad\text{et}\quad V=\frac{-tt\sqrt{p}}{2(\gamma-\delta)}.$$

XII

Hinc ergo adipiscimur sequentem aequationem integratam

$$\int\frac{xxdx}{\sqrt{(A+2Bx+Cxx)}}-\int\frac{yydy}{\sqrt{(A+2By+Cyy)}}=\text{Const.}-\frac{(x+y)^2\sqrt{p}}{2(\gamma-\delta)}$$

atque in genere concludimus fore

$$\int\frac{dx(\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}xx)}{\sqrt{(A+2Bx+Cxx)}}-\int\frac{dy(\mathfrak{A}+\mathfrak{B}y+\mathfrak{C}yy)}{\sqrt{(A+2By+Cyy)}}=\text{Const.}-\frac{\mathfrak{B}(x+y)\sqrt{p}}{\gamma-\delta}-\frac{\mathfrak{C}(x+y)^2\sqrt{p}}{2(\gamma-\delta)},$$

siquidem fuerit $0=\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy$. Erit autem ex

relationibus supra assignatis

$$\frac{\sqrt{p}}{\gamma-\delta}=\frac{-\beta}{B\sqrt{p}} \quad \text{sive} \quad \frac{\sqrt{p}}{\gamma-\delta}=-\sqrt{\frac{2A\beta-B\alpha}{B(2B\beta-C\alpha)}}.$$

XIII

Ponatur iam in genere

$$\frac{x^n dx}{\sqrt{(A+2Bx+Cxx)}}-\frac{y^n dy}{\sqrt{(A+2By+Cyy)}}=dV$$

eritque ponendo

$$T=\beta\beta+\beta(\gamma+\delta)t+\gamma\delta tt+(\gamma-\delta)^2u$$

$$x^n dx(\beta+\gamma x+\delta y)+y^n dy(\beta+\gamma y+\delta x)=\frac{TdV}{\sqrt{p}},$$

at ob $x+y=t$ et $xy=u$ habebimus

$$x=\frac{t+\sqrt{(tt-4u)}}{2} \quad \text{et} \quad y=\frac{t-\sqrt{(tt-4u)}}{2}$$

ideoque

$$\beta+\gamma x+\delta y=\frac{2\beta+(\gamma+\delta)t+(\gamma-\delta)\sqrt{(tt-4u)}}{2},$$

$$\beta+\gamma y+\delta x=\frac{2\beta+(\gamma+\delta)t-(\gamma-\delta)\sqrt{(tt-4u)}}{2}.$$

XIV

Differentiando autem habebimus

$$dx=\frac{dt\sqrt{(tt-4u)}+tdt-2du}{2\sqrt{(tt-4u)}} \quad \text{et} \quad dy=\frac{dt\sqrt{(tt-4u)}-tdt+2du}{2\sqrt{(tt-4u)}},$$

at ante vidimus esse $du=\frac{dt(\beta+\gamma t)}{\gamma-\delta}$; quo valore substituto prodibit

$$dx=\frac{-dt(2\beta+(\gamma+\delta)t-(\gamma-\delta)\sqrt{(tt-4u)})}{2(\gamma-\delta)\sqrt{(tt-4u)}},$$

$$dy=\frac{dt(2\beta+(\gamma+\delta)t+(\gamma-\delta)\sqrt{(tt-4u)})}{2(\gamma-\delta)\sqrt{(tt-4u)}}$$

hisque valoribus substitutis

$$dx(\beta+\gamma x+\delta y)=\frac{-dt(4\beta\beta+4\beta(\gamma+\delta)t+4\gamma\delta tt+4(\gamma-\delta)^2u)}{4(\gamma-\delta)\sqrt{(tt-4u)}}=\frac{-Tdt}{(\gamma-\delta)\sqrt{(tt-4u)}}$$

et

$$dy(\beta+\gamma y+\delta x)=\frac{+Tdt}{(\gamma-\delta)\sqrt{(tt-4u)}}.$$

XV

Nostra ergo aequatione per T divisa habebimus

$$\frac{-dt(x^n-y^n)}{(\gamma-\delta)\sqrt{(tt-4u)}}=\frac{dV}{\sqrt{p}}\quad\text{et}\quad V=\frac{-\sqrt{p}}{\gamma-\delta}\int\frac{dt(x^n-y^n)}{\sqrt{(tt-4u)}}$$

existente

$$x=\frac{t+\sqrt{(tt-4u)}}{2}\quad\text{et}\quad y=\frac{t-\sqrt{(tt-4u)}}{2}$$

atque

$$u=\frac{\alpha+2\beta t+\gamma tt}{2(\gamma-\delta)};\quad\text{unde}\quad \sqrt{(tt-4u)}=\sqrt{\frac{2\alpha+4\beta t+(\gamma+\delta)tt}{\delta-\gamma}}.$$

Unde valores ipsius $\frac{x^n-y^n}{\sqrt{(tt-4u)}}$ ex sequente progressione colligi poterunt:

$$\frac{x^0-y^0}{\sqrt{(tt-4u)}}=0,$$

$$\frac{x^1-y^1}{\sqrt{(tt-4u)}}=1,$$

$$\frac{x^2-y^2}{\sqrt{(tt-4u)}}=t,$$

$$\frac{x^3-y^3}{\sqrt{(tt-4u)}}=tt-u=\frac{(\gamma-2\delta)tt-2\beta t-\alpha}{2(\gamma-\delta)},$$

$$\frac{x^4-y^4}{\sqrt{(tt-4u)}}=t^3-2tu=\frac{-2\delta t^3-4\beta tt-2\alpha t}{2(\gamma-\delta)},$$

$$\frac{x^5-y^5}{\sqrt{(tt-4u)}}=t^4-3ttu+uu$$

$$=\frac{-(\gamma\gamma+2\gamma\delta-4\delta\delta)t^4-4\beta(2\gamma-3\delta)t^3+(4\beta\beta-4\alpha\gamma+6\alpha\delta)tt+4\alpha\beta t+\alpha\alpha}{4(\gamma-\delta)^2},$$

etc.

XVI

Nanciscemur ergo formulas sequentes integratas

$$\int \frac{x^3 dx}{\sqrt{(A+2Bx+Cxx)}} - \int \frac{y^3 dy}{\sqrt{(A+2By+Cyy)}}$$
$$= \text{Const.} - \frac{\sqrt{p}}{2(\gamma-\delta)^3}\left(\frac{1}{3}(\gamma-2\delta)(x+y)^3 - \beta(x+y)^2 - \alpha(x+y)\right),$$

$$\int \frac{x^4 dx}{\sqrt{(A+2Bx+Cxx)}} - \int \frac{y^4 dy}{\sqrt{(A+2By+Cyy)}}$$
$$= \text{Const.} + \frac{\sqrt{p}}{(\gamma-\delta)^2}\left(\frac{1}{4}\delta(x+y)^4 + \frac{2}{3}\beta(x+y)^3 + \frac{1}{2}\alpha(x+y)^2\right),$$

quae scilicet locum habent, si variabiles x et y ita a se invicem pendent, ut sit $0 = \alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy$ atque hi coefficientes pariter atque p secundum praescriptas formulas ex datis A, B, C determinentur.

XVII

Hinc ergo infinitae formulae integrales exhiberi possunt, quae etsi ipsae non sint integrabiles, earum tamen differentia vel sit constans vel geometrice seu algebraice assignari queat. Quae comparatio, cum in Analysi insignem habeat usum, tum imprimis in arcubus curvarum irrectificabilium inter se comparandis summam affert utilitatem, quam in aliquot exemplis ostendisse iuvabit.

DE COMPARATIONE ARCUUM CIRCULI

1. Sit radius circuli $=1$ in eoque abscissa a centro sumta $=z$; erit arcus ei respondens $=\int \frac{dz}{\sqrt{(1-zz)}}$, cuius propterea sinus est $=z$. Ut igitur nostrae formulae huiusmodi arcus circuli exprimant, poni debet $A=1$, $B=0$, $C=-1$; quo facto habebimus

$$\beta\beta - \alpha\gamma = p, \quad \beta(\delta-\gamma) = 0 \quad \text{et} \quad \delta\delta - \gamma\gamma = -p;$$

has enim determinationes ab ipsa origine peti oportet, quia ob $B=0$ valores inventi fiunt incongrui. Iam ex formula secunda sequitur vel $\delta - \gamma = 0$ vel $\beta = 0$, quorum ille valor $\delta = \gamma$ formulae tertiae adversatur. Erit ergo $\beta = 0$,

$= \pm \sqrt{(\gamma\gamma - p)}$ et $\alpha = \frac{-p}{\gamma}$. Ambae ergo quantitates constantes γ et p rbitrio nostro relinquuntur.

2. Quo formulae nostrae fiant simpliciores, ponamus $\gamma = 1$ et $p = cc$ ritque

$$\alpha = -cc, \quad \beta = 0, \quad \gamma = 1 \quad \text{et} \quad \delta = -\sqrt{(1-cc)}$$

c nostra aequatio canonica relationem variabilium x et y determinans fiet

$$0 = -cc + xx + yy - 2xy\sqrt{(1-cc)},$$

x qua colligitur

$$y = x\sqrt{(1-cc)} \pm c\sqrt{(1-xx)}.$$

3. Quodsi ergo iste valor ipsi y tribuatur, erit

$$\int \frac{dx}{\sqrt{(1-xx)}} - \int \frac{dy}{\sqrt{(1-yy)}} = \text{Const.}$$

Denotemus brevitatis gratia haec integralia ita

$$\int \frac{dx}{\sqrt{(1-xx)}} = \Pi.x \quad \text{et} \quad \int \frac{dy}{\sqrt{(1-yy)}} = \Pi.y$$

atque $\Pi.x$ et $\Pi.y$ indicabunt arcus circuli abscissis seu sinibus x et y respondentes. Quocirca erit

$$\Pi.x - \Pi.\big(x\sqrt{(1-cc)} + c\sqrt{(1-xx)}\big) = \text{Const.}$$

4. Ad constantem determinandam ponatur $x = 0$ et ob $\Pi.0 = 0$ fiet Const. $= -\Pi.c$ sicque erit

$$\Pi.c + \Pi.x = \Pi.\big(x\sqrt{(1-cc)} + c\sqrt{(1-xx)}\big);$$

... aequalis summae duorum arcuum quorumcunque.

5. Si in formula priori ponatur $x = c$, erit

$$2\Pi . c = \Pi . 2c\sqrt{(1-cc)}.$$

Ac si porro ponatur $x = 2c\sqrt{(1-cc)}$, ut sit $\Pi . x = 2\Pi . c$, erit ob $\sqrt{(1-xx)} = 1-2cc$

$$3\Pi . c = \Pi . (3c - 4c^3).$$

Posito autem ultra $x = 3c - 4c^3$ erit

$$4\Pi . c = \Pi . \big(x\sqrt{(1-cc)} + c\sqrt{(1-xx)}\big),$$

unde multiplicatio arcuum circularium est manifesta.

DE COMPARATIONE ARCUUM PARABOLAE

6. Existente AB (Fig. 1) parabolae axe sumentur abscissae AP in tangente verticis A sitque parameter parabolae $= 2$; unde vocata abscissa quacunque $AP = z$ erit applicata $Pp = \frac{zz}{2}$ ideoque arcus $Ap = \int dz\sqrt{(1+zz)}$; quae expressio ut ad nostras formulas reducatur, in hanc abit $\int \frac{dz(1+zz)}{\sqrt{(1+zz)}}$. Quare fieri oportet $A = 1$, $B = 0$ et $C = 1$, unde ut ante habebimus

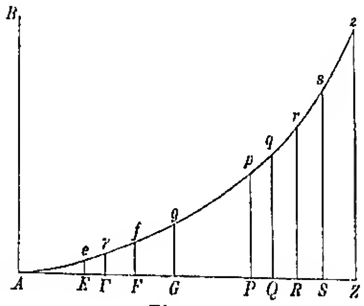

Fig. 1.

$$\beta = 0, \quad \alpha = {}^{-}p \quad \text{et} \quad \delta = \pm\sqrt{(\gamma\gamma + p)}.$$

Sit ergo $\gamma = 1$ et $p = cc$ atque aequatio relationem inter x et y exhibens erit

$$0 = -cc + xx + yy - 2xy\sqrt{(cc+1)}$$

seu

$$y = x\sqrt{(1+cc)} + c\sqrt{(1+xx)}.$$

7. Deinde ob $\sqrt{p} = c$ et $\gamma - \delta = 1 + \sqrt{(1+cc)}$ facto $\mathfrak{A} = 1$, $\mathfrak{B} = 0$ et $\mathfrak{C} = 1$ erit ex formula XII data

$$\int \frac{dx(1+xx)}{\sqrt{(1+xx)}} - \int \frac{dy(1+yy)}{\sqrt{(1+yy)}} = \text{Const.} - \frac{c(x+y)^2}{2+2\sqrt{(1+cc)}}.$$

At est
$$x + y = x\left(1 + \sqrt{(1 + cc)}\right) + c\sqrt{(1 + xx)},$$
ergo
$$(x + y)^2 = 2xx\left(1 + cc + \sqrt{(1 + cc)}\right) + cc + 2cx\left(1 + \sqrt{(1 + cc)}\right)\sqrt{(1 + xx)}.$$
Quare formularum istarum integralium differentia erit
$$\text{Const.} - cxx\sqrt{(1 + cc)} - ccx\sqrt{(1 + xx)} = \text{Const.} - cxy.$$

8. Indicetur arcus parabolae abscissae cuicunque z respondens $\int dz\sqrt{(1 + zz)}$ per $\Pi . z$ et nostra aequatio hanc induet formam
$$\Pi . x - \Pi . \left(x\sqrt{(1 + cc)} + c\sqrt{(1 + xx)}\right) = -\Pi . c - cx\left(x\sqrt{(1 + cc)} + c\sqrt{(1 + xx)}\right)$$
sive
$$\Pi . c + \Pi . x = \Pi . \left(x\sqrt{(1 + cc)} + c\sqrt{(1 + xx)}\right) - cx\left(x\sqrt{(1 + cc)} + c\sqrt{(1 + xx)}\right).$$
Datis ergo duobus arcubus quibuscunque tertius arcus assignari potest, qui a summa illorum deficiat quantitate geometrice assignabili. Vel quo indoles huius aequationis clarius perspiciatur, erit
$$\Pi . c + \Pi . x = \Pi . y - cxy,$$
siquidem fuerit
$$y = x\sqrt{(1 + cc)} + c\sqrt{(1 + xx)}.$$

9. Cum sit $y > x$, sint in figura abscissae $AE = c$, $AF = x$ et $AG = y$; erit arcus $Ae = \Pi . c$ et arcus $fg = \Pi . y - \Pi . x$; hinc ergo habebimus
$$\text{Arc.}\, Ae = \text{Arc.}\, fg - cxy \quad \text{seu} \quad \text{Arc.}\, fg - \text{Arc.}\, Ae = cxy$$
existente $y = x\sqrt{(1 + cc)} + c\sqrt{(1 + xx)}$. Ex his igitur sequentia problemata circa parabolam resolvi poterunt.

PROBLEMA 1

10. *Dato arcu parabolae Ae in vertice A terminato a puncto quovis f alium abscindere arcum fg, ita ut differentia horum arcuum $fg - Ae$ geometrice assignari queat.*

SOLUTIO

Ponatur arcus dati Ae abscissa $AE = e$ et abscissa termino dato f arcus quaesiti fg respondens $AF = f$, abscissa vero alteri termino g arcus quaesiti respondens $AG = g$, quae ita accipiatur, ut sit

$$g = f\sqrt{(1+ee)} + e\sqrt{(1+ff)},$$

eritque existente parabolae parametro $= 2$, uti constanter assumemus,

$$\text{Arc.}\, fg - \text{Arc.}\, Ae = efg.$$

A puncto autem f quoque retrorsum arcus abscindi potest $f\gamma$, qui superet arcum Ae quantitate algebraica; ob signum radicale $\sqrt{(1+ff)}$ enim ambiguum capiatur

$$A\Gamma = \gamma = f\sqrt{(1+ee)} - e\sqrt{(1+ff)}$$

eritque

$$\text{Arc.}\, f\gamma - \text{Arc.}\, Ae = ef\gamma.$$

Q. E. I.

COROLLARIUM 1

11. Inventis ergo his duobus punctis g et γ erit quoque arcuum fg et $f\gamma$ differentia geometrice assignabilis; erit enim

$$\text{Arc.}\, fg - \text{Arc.}\, f\gamma = ef(g-\gamma).$$

At est

$$g - \gamma = 2e\sqrt{(1+ff)},$$

unde $e = \frac{g-\gamma}{2\sqrt{(1+ff)}}$. Tum vero habemus

$$g + \gamma = 2f\sqrt{(1+ee)}$$

sive $\sqrt{(1+ee)} = \frac{g+\gamma}{2f}$; unde eliminanda e fit

$$1 = \frac{(g+\gamma)^2}{4ff} - \frac{(g-\gamma)^2}{4(1+ff)}$$

seu

$$4ff(1+ff) = (g+\gamma)^2 + 4ffg\gamma.$$

Fit ergo

$$\gamma = -g(1+2ff) + 2f\sqrt{(1+ff)(1+gg)}.$$

COROLLARIUM 2

12. Dato ergo arcu quocunque fg existente $AF = f$ et $AG = g$ a puncto f retrorsum arcus $f\gamma$ abscindi potest, ita ut arcuum fg et $f\gamma$ differentia fiat geometrica. Capiatur scilicet $A\Gamma = \gamma = -g(1+2ff) + 2f\sqrt{(1+ff)(1+gg)}$ eritque

$$\text{Arc.}\, fg - \text{Arc.}\, f\gamma = 2f\left(g\sqrt{(1+ff)} - f\sqrt{(1+gg)}\right)^2\sqrt{(1+ff)}.$$

Horum ergo arcuum differentia evanescere nequit, nisi sit vel $f=0$, quo casu fit $\gamma = -g$, vel $g = f$, quo casu uterque arcus fg et $f\gamma$ evanescit.

COROLLARIUM 3

13. Ut igitur positis $AE = e$, $AF = f$, $AG = g$ differentia arcuum fg et Ae fiat geometrice assignabilis, scilicet $\text{Arc.}\, fg - \text{Arc.}\, Ae = efg$, oportet sit

$$g = f\sqrt{(1+ee)} + e\sqrt{(1+ff)},$$

seu ex trium quantitatum e, f, g binis datis tertia ita determinatur, ut sit vel

$$g = f\sqrt{(1+ee)} + e\sqrt{(1+ff)}$$

vel

$$f = g\sqrt{(1+ee)} - e\sqrt{(1+gg)}$$

vel

$$e = g\sqrt{(1+ff)} - f\sqrt{(1+gg)}.$$

COROLLARIUM 4

14. Cum sit $g = f\sqrt{(1+ee)} + e\sqrt{(1+ff)}$, erit

$$\sqrt{(1+gg)} = ef + \sqrt{(1+ee)(1+ff)},$$

unde colligitur

$$g + \sqrt{(1+gg)} = \left(e + \sqrt{(1+ee)}\right)\left(f + \sqrt{(1+ff)}\right).$$

Ergo ut arcus fg superet arcum Ae quantitate algebraica efg, oportet, ut sit

$$\frac{g+\sqrt{(1+gg)}}{f+\sqrt{(1+ff)}} = e + \sqrt{(1+ee)}.$$

COROLLARIUM 5

15. Haec ultima formula ideo est notatu digna, quod in ea quantitatum e, f et g functiones sint a se invicem separatae. Quodsi ergo ponatur

$$e+\sqrt{(1+ee)}=E,\quad f+\sqrt{(1+ff)}=F,\quad g+\sqrt{(1+gg)}=G,$$

erit

$$e=\frac{EE-1}{2E},\quad f=\frac{FF-1}{2F},\quad g=\frac{GG-1}{2G}.$$

Quare si capiatur $\frac{G}{F}=E$, erit arcuum differentia

$$\text{Arc.}\,fg-\text{Arc.}\,Ae=efg=\frac{(EE-1)(FF-1)(GG-1)}{8EFG}$$

seu

$$\text{Arc.}\,fg-\text{Arc.}\,Ae=\frac{(FF-1)(GG-1)(GG-FF)}{8FFGG}=\frac{fg(GG-FF)}{2FG}.$$

PROBLEMA 2

16. *Dato arcu parabolae quocunque fg a puncto parabolae dato p alium abscindere arcum pq ita, ut differentia horum duorum arcuum fg et pq fiat geometrice assignabilis.*

SOLUTIO

Pro arcu dato fg ponantur abscissae $AF=f$, $AG=g$; pro arcu autem quaesito pq sint abscissae $AP=p$, $AQ=q$. Iam a vertice parabolae concipiatur arcus Ae respondens abscissae $AE=e$, cuius defectus ab utroque illorum arcuum sit geometrice assignabilis. Ad hoc autem vidimus (§ 14) requiri, ut sit

$$\frac{g+\sqrt{(1+gg)}}{f+\sqrt{(1+ff)}}=e+\sqrt{(1+ee)}\quad\text{et}\quad\frac{q+\sqrt{(1+qq)}}{p+\sqrt{(1+pp)}}=e+\sqrt{(1+ee)}.$$

Ponamus brevitatis gratia

$$f+\sqrt{(1+ff)}=F,\qquad p+\sqrt{(1+pp)}=P,$$
$$g+\sqrt{(1+gg)}=G,\qquad q+\sqrt{(1+qq)}=Q$$

tque ut problemati satisfiat, necesse est sit $\frac{G}{F}=\frac{Q}{P}$. Porro autem cum sit x § 15

$$\text{Arc.}\,fg - \text{Arc.}\,Ae = \frac{fg(GG-FF)}{2FG}$$

imiliterque

$$\text{Arc.}\,pq - \text{Arc.}\,Ae = \frac{pq(QQ-PP)}{2PQ},$$

rit arcuum determinatorum differentia

$$\text{Arc.}\,pq - \text{Arc.}\,fg = \frac{pq(QQ-PP)}{2PQ} - \frac{fg(GG-FF)}{2FG}$$

deoque geometrico assignabilis. Q. E. I.

COROLLARIUM 1

17. Cum autem sit $\frac{G}{F}=\frac{Q}{P}$, erit

$$\frac{QQ-PP}{2PQ} = \frac{GG-FF}{2FG},$$

unde differentia arcuum determinatorum prodit

$$\text{Arc.}\,pq - \text{Arc.}\,fg = \frac{(pq-fg)(GG-FF)}{2FG}.$$

Est autem

$$f=\frac{FF-1}{2F},\quad g=\frac{GG-1}{2G},\quad p=\frac{PP-1}{2P},\quad q=\frac{QQ-1}{2Q}$$

ideoque ob $Q=\frac{GP}{F}$ erit

$$q=\frac{GGPP-FF}{2FGP}.$$

COROLLARIUM 2

18. Erit ergo

$$pq=\frac{(PP-1)(GGPP-FF)}{4FGPP} \quad \text{et} \quad fg=\frac{(FF-1)(GG-1)}{4FG}$$

ideoque

$$pq-fg=\frac{(PP-FF)(GGPP-1)}{4FGPP}.$$

Hinc arcuum differentia prodit

$$\text{Arc.}\,pq - \text{Arc.}\,fg = \frac{(GG-FF)(PP-FF)(GGPP-1)}{8FFGGPP}.$$

COROLLARIUM 3

19. Ut igitur arcus pq arcui fg adeo fiat aequalis, esse oportet vel $GG - FF = 0$ vel $PP - FF = 0$ vel $GGPP - 1 = 0$. Primo autem casu arcus fg ideoque et pq evanescit; altero casu punctum p in f ideoque et q in g cadit arcusque ergo pq non prodit diversus ab arcu fg; tertius autem casus dat $P = \frac{1}{G}$ seu

$$p + \sqrt{(1+pp)} = \frac{1}{g+\sqrt{(1+gg)}} = \sqrt{(1+gg)} - g,$$

unde fit $p = -g$ et $q = -f$, ita ut pq in alterum ramum parabolae cadat arcuique fg similis et aequalis prodeat.

COROLLARIUM 4

20. Hinc ergo sequitur in parabola non exhiberi posse duos arcus dissimiles, qui sint inter se aequales. Interim proposito quocunque arcu fg infinitis modis alius abscindi potest pq, qui illum quantitate algebraica superet vel ab eo deficiat. Superabit scilicet, si fuerit $P > F$ seu $AP > AF$; deficiet autem, si $P < F$ seu $AP < AF$.

PROBLEMA 3

21. *Dato parabolae arcu quocunque fg a dato puncto p alium arcum abscindere pr, qui duplum arcus fg superet quantitate geometrice assignabili.*

SOLUTIO

Positis ut ante abscissis $AF = f$, $AG = g$, $AP = p$, $AQ = q$ sit $AR = r$ denotentque litterae maiusculae F, G, P, Q, R istas functiones $f + \sqrt{(1+ff)}$, $g + \sqrt{(1+gg)}$ etc. minuscularum cognominum. Primum igitur si statuatur $\frac{Q}{P} = \frac{G}{F}$, erit

$$\text{Arc.}\, pq - \text{Arc.}\, fg = \frac{(pq-fg)(GG-FF)}{2FG}.$$

nili autem modo si statuatur $\frac{R}{Q} = \frac{G}{F}$, erit

$$\text{Arc.}\, qr - \text{Arc.}\, fg = \frac{(qr-fg)(GG-FF)}{2FG}.$$

ldantur ergo invicem hae duae aequationes; erit

$$\text{Arc.}\, pr - 2\,\text{Arc.}\, fg = \frac{(pq+qr-2fg)(GG-FF)}{2FG}.$$

t iam ex calculo eliminentur litterae q et Q, erit primo $\frac{R}{P} = \frac{GG}{FF}$; tum vero est

$$q = \frac{GGPP-FF}{2FGP} \quad \text{seu} \quad q = \frac{F(PR-1)}{2GP}$$

t ob

$$p = \frac{PP-1}{2P} \quad \text{et} \quad r = \frac{G^4P^2-F^4}{2F^2G^2P}$$

rit

$$p + r = \frac{(FF+GG)(GGPP-FF)}{2FFGGP}$$

deoque

$$pq + qr = \frac{(FF+GG)(GGPP-FF)^2}{4F^3G^3PP} \quad \text{et} \quad 2fg = \frac{2(FF-1)(GG-1)}{4FG}.$$

Sumto ergo $\frac{R}{P} = \frac{GG}{FF}$ arcus pr superabit duplum arcus fg quantitate algebraica. Q. E. I.

COROLLARIUM 1

22. Punctum igitur p ita assumi poterit, ut excessus arcus pr supra duplum arcum $2fg$ sit datae magnitudinis; definietur enim P per aequationem algebraicam ope extractionis radicis quadratae tantum.

COROLLARIUM 2

23. Fieri igitur poterit, ut arcus pr praecise sit duplus arcus dati fg, quod evenit, si P definiatur ex hac aequatione

$$(GGPP-FF)^2 = \frac{2(FF-1)(GG-1)FFGGPP}{FF+GG},$$

unde elicitur

$$\frac{GGPP}{FF} = \frac{FFGG + 1 + \sqrt{(F^4 - 1)(G^4 - 1)}}{FF + GG}$$

et

$$\frac{GP}{F} = \frac{\sqrt{\tfrac{1}{2}(FF+1)(GG+1)} + \sqrt{\tfrac{1}{2}(FF-1)(GG-1)}}{\sqrt{(FF+GG)}} = \frac{FR}{G}.$$

COROLLARIUM 3

24. Haec autem determinatio arcus dupli pr maxime fit obvia, si arcus datus fg in vertice A incipiat; tum enim ob $F = 1$ fit $GP = F$ seu $P = \frac{1}{G} = \sqrt{(1 + gg)} - g$. Obtinetur ergo $p = -g$ et $R = G$ ideoque $r = g$. Hoc scilicet casu arcus pr in parabola circa verticem A utrinque aequaliter extendetur sicque manifesto fit duplus arcus propositi.

COROLLARIUM 4

25. Fieri quoque potest, ut arcus pr in ipso puncto g terminetur sicque ambo arcus, simplus fg et duplus pr, evadant contigui. Hoc nempe evenit, si $P = G$, quo casu haec habetur aequatio

$$F^6 + F^4G^2 - 2F^4G^6 + F^2G^8 - 2F^2G^4 + G^{10} = 0,$$

quae per $FF - GG$ divisa praebet

$$F^4 - 2FFG^6 + 2FFGG - G^8 = 0,$$

unde elicitur

$$FF = GG(G^4 - 1) + GG\sqrt{(G^8 - G^4 + 1)}$$

ideoque

$$F = G\sqrt{\left(G^4 - 1 + \sqrt{(G^8 - G^4 + 1)}\right)}$$

et

$$R = \frac{G^3}{FF} \quad \text{seu} \quad R = \frac{\sqrt{(G^8 - G^4 + 1)} - G^4 + 1}{G^3}.$$

COROLLARIUM 5

26. Quantitas ergo G seu parabolae punctum g pro lubitu assumi licet, in quo duo arcus terminabuntur, quorum alter alterius exacte erit duplus. Cum autem sumto g affirmativo ideoque $G > 1$ prodeat $F > G$, punctum f

a vertice magis erit remotum quam punctum g; tum vero reperitur

$$r = \frac{RR-1}{2R} = \frac{-(GG-1)\sqrt{(G^8-G^4+1)}-G^6-G^4+GG+1}{2G^3};$$

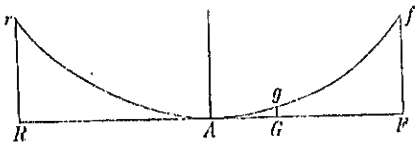

Fig. 2.

cuius valor cum sit negativus, punctum r in alterum parabolae ramum incidit. Arcus ergo ita erunt dispositi, ut habet figura 2, eritque

$$\text{Arc.}\, gr = 2\, \text{Arc.}\, fg.$$

COROLLARIUM 6

27. Sit g valde parvum; erit $G = 1 + g + \frac{1}{2}gg$ hincque

$$G^2 = 1 + 2g + 2gg, \quad G^3 = 1 + 3g + \tfrac{9}{2}gg, \quad G^4 = 1 + 4g + 8gg$$

et

$$G^8 = 1 + 8g + 32gg,$$

unde

$$F = \left(1 + g + \tfrac{1}{2}gg\right)\left(1 + 3g + \tfrac{9}{2}gg\right) = 1 + 4g + 8gg,$$

ergo $f = \frac{FF-1}{2F} = 4g$; porro $R = 1 - 5g + \frac{25}{2}gg$, unde $r = -5g$. Quare (Fig. 2) si $Ag = g$ valde parvum, erit proxime $AF = 4AG$ et $AR = 5AG$, ita ut sit quoque $GR = 2GF$.

SCHOLION

28. Antequam ad ulteriorem arcuum parabolicorum multiplicationem progrediamur, etiamsi ea ex formulis datis non difficulter erui queat, tamen expediet differentiam algebraicam arcuum parabolicorum commodius exprimere. Cum igitur (Fig. 1, p. 306) positis abscissis $AE = e$, $AF = f$, $AG = g$ invenerimus (§ 13) $\text{Arc.}\, Ag - \text{Arc.}\, Af - \text{Arc.}\, Ae = efg$ existente $e = g\sqrt{(1+ff)} - f\sqrt{(1+gg)}$, videndum est, num quantitas efg non possit transformari in terna membra, quae sint singula functiones certae ipsarum e, f et g, ita ut sit

$$efg = \text{funct.}\, g - \text{funct.}\, f - \text{funct.}\, e;$$

sic enim quaelibet harum functionum cum arcu cognomine comparari posset. Cum autem sit

$$efg = fgg\sqrt{(1+ff)} - ffg\sqrt{(1+gg)} \quad \text{et} \quad \sqrt{(1+ee)} = \sqrt{(1+ff)(1+gg)} - fg,$$

erit

$$e\sqrt{(1+ee)} = g\sqrt{(1+gg)} + 2ffg\sqrt{(1+gg)} - f\sqrt{(1+ff)} - 2fgg\sqrt{(1+ff)}$$

hincque

$$fgg\sqrt{(1+ff)} - ffg\sqrt{(1+gg)} = efg = \frac{1}{2}g\sqrt{(1+gg)} - \frac{1}{2}f\sqrt{(1+ff)} - \frac{1}{2}e\sqrt{(1+ee)},$$

quae est expressio talis, qualis desideratur. Quare si istas abscissarum e, f, g functiones brevitatis gratia ponamus

$$\frac{1}{2}e\sqrt{(1+ee)} = \mathfrak{E}, \quad \frac{1}{2}f\sqrt{(1+ff)} = \mathfrak{F} \quad \text{et} \quad \frac{1}{2}g\sqrt{(1+gg)} = \mathfrak{G},$$

habebimus

$$\text{Arc.}\, Ag - \text{Arc.}\, Af - \text{Arc.}\, Ae = \mathfrak{G} - \mathfrak{F} - \mathfrak{E} = \text{Arc.}\, fg - \text{Arc.}\, Ae.$$

Si porro has functiones cum illis, quibus ante usi sumus, comparemus, scilicet

$$e + \sqrt{(1+ee)} = E, \quad f + \sqrt{(1+ff)} = F, \quad g + \sqrt{(1+gg)} = G,$$

erit

$$\mathfrak{E} = \frac{E^4-1}{8EE}, \quad \mathfrak{F} = \frac{F^4-1}{8FF}, \quad \mathfrak{G} = \frac{G^4-1}{8GG}$$

et ex natura horum arcuum est $\frac{G}{F} = E$. Si iam simili modo pro arcu pq procedamus et ex abscissis $AP = p$ et $AQ = q$ has formemus functiones

$$p + \sqrt{(1+pp)} = P, \qquad \frac{1}{2}p\sqrt{(1+pp)} = \mathfrak{P},$$

$$q + \sqrt{(1+qq)} = Q, \qquad \frac{1}{2}q\sqrt{(1+qq)} = \mathfrak{Q},$$

erit simili modo

$$\text{Arc.}\, pq - \text{Arc.}\, Ae = \mathfrak{Q} - \mathfrak{P} - \mathfrak{E}$$

existente $\frac{Q}{P} = E$. Hinc si illa aequatio ab hac subtrahatur, remanebit

$$\text{Arc.}\,pq - \text{Arc.}\,fg = (\mathfrak{Q} - \mathfrak{P}) - (\mathfrak{G} - \mathfrak{F}),$$

si modo fuerit $\frac{Q}{P} = \frac{G}{F}$.

PROBLEMA 4

29. *Dato arcu parabolae quocunque fg abscindere arcum alium pz, qui ad arcum fg sit in data ratione $n:1$.*

SOLUTIO

Positis abscissis $AF = f$, $AG = g$ capiantur plures abscissae $AP = p$, $AQ = q$, $AR = r$, $AS = s$ et ultima $AZ = z$, ex quibus formentur geminae functiones litteris maiusculis cum latinis tum germanicis cognominibus denotandae, scilicet

$$f + \sqrt{(1+ff)} = F, \qquad g + \sqrt{(1+gg)} = G, \qquad p + \sqrt{(1+pp)} = P \quad \text{etc.}$$

$$\frac{1}{2} f\sqrt{(1+ff)} = \mathfrak{F}, \qquad \frac{1}{2} g\sqrt{(1+gg)} = \mathfrak{G}, \qquad \frac{1}{2} p\sqrt{(1+pp)} = \mathfrak{P} \quad \text{etc.}$$

sitque primo $\frac{Q}{P} = \frac{G}{F}$; erit

$$\text{Arc.}\,pq - \text{Arc.}\,fg = (\mathfrak{Q} - \mathfrak{P}) - (\mathfrak{G} - \mathfrak{F}).$$

Deinde sit $\frac{R}{Q} = \frac{G}{F}$ seu $\frac{R}{P} = \frac{G^2}{F^2}$; erit

$$\text{Arc.}\,qr - \text{Arc.}\,fg = (\mathfrak{R} - \mathfrak{Q}) - (\mathfrak{G} - \mathfrak{F}),$$

qua aequatione ad priorem addita fit

$$\text{Arc.}\,pr - 2\,\text{Arc.}\,fg = (\mathfrak{R} - \mathfrak{P}) - 2(\mathfrak{G} - \mathfrak{F}).$$

Sit porro $\frac{S}{R} = \frac{G}{F}$ seu $\frac{S}{P} = \frac{G^3}{F^3}$; erit

$$\text{Arc.}\,rs - \text{Arc.}\,fg = (\mathfrak{S} - \mathfrak{R}) - (\mathfrak{G} - \mathfrak{F}),$$

qua iterum ad praecedentem adiecta obtinebitur

$$\text{Arc.}\,ps - 3\,\text{Arc.}\,fg = (\mathfrak{S} - \mathfrak{P}) - 3(\mathfrak{G} - \mathfrak{F}).$$

Simili modo si ulterius ponatur $\frac{T}{S}=\frac{G}{F}$ seu $\frac{T}{P}=\frac{G^4}{F^4}$, erit

$$\text{Arc.}\,pt-4\,\text{Arc.}\,fg=(\mathfrak{T}-\mathfrak{P})-4(\mathfrak{G}-\mathfrak{F}).$$

Unde perspicitur, si z sit ultimum punctum arcus pz, qui quaeritur, et posita $AZ=z$ sit

$$Z=z+\sqrt{(1+zz)}\quad \text{et}\quad \mathfrak{Z}=\frac{1}{2}z\sqrt{(1+zz)},$$

poni debere $\frac{Z}{P}=\frac{G^n}{F^n}$ tumque fore

$$\text{Arc.}\,pz-n\,\text{Arc.}\,fg=(\mathfrak{Z}-\mathfrak{P})-n(\mathfrak{G}-\mathfrak{F}).$$

Nunc ut sit $\text{Arc.}\,pz=n\,\text{Arc.}\,fg$, reddi oportet $\mathfrak{Z}-\mathfrak{P}=n(\mathfrak{G}-\mathfrak{F})$. At est

$$\mathfrak{Z}=\frac{Z^4-1}{8ZZ},\quad \mathfrak{P}=\frac{P^4-1}{8PP},\quad \mathfrak{G}=\frac{G^4-1}{8GG}\quad \text{et}\quad \mathfrak{F}=\frac{F^4-1}{8FF}.$$

Verum ob $Z=\frac{G^nP}{F^n}$ erit

$$\mathfrak{Z}=\frac{G^{4n}P^4-F^{4n}}{8F^{2n}G^{2n}PP}.$$

Quibus valoribus substitutis sequens acquiretur aequatio resolvenda

$$\frac{G^{4n}P^4-F^{4n}}{F^{2n}G^{2n}PP}=\frac{P^4-1}{PP}+\frac{n(GG-FF)(1+FFGG)}{FFGG}$$

sive

$$0=G^{2n}(G^{2n}-F^{2n})P^4+F^{2n}(G^{2n}-F^{2n})-nF^{2n-2}G^{2n-2}(G^2-F^2)(F^2G^2+1)PP$$

seu

$$P^4=\frac{nF^{2n}(G^2-F^2)(F^2G^2+1)P^2}{F^2G^2(G^{2n}-F^{2n})}-\frac{F^{2n}}{G^{2n}}.$$

Quocunque ergo assumto multiplicationis indice n, sive numero integro sive fracto, ex hac aequatione semper definiri potest P, unde arcus quaesiti pz alter terminus p innotescit. Quo invento pro altero termino z erit $Z=\frac{G^nP}{F^n}$ sicque obtinebitur arcus pz, ut sit $pz=n\cdot fg$. Q. E. I.

COROLLARIUM 1

30. Si loco P quaerere velimus Z, in ultima aequatione substitui oportet $\frac{F^nZ}{G^n}$ prodibitque

$$Z^4 = \frac{nG^{2n}(G^2-F^2)(F^2G^2+1)ZZ}{F^2G^2(G^{2n}-F^{2n})} - \frac{G^{2n}}{F^{2n}},$$

ubi litterae F et G pariter uti P et Z sunt inter se commutatae.

COROLLARIUM 2

31. Cum $G^{2n}-F^{2n}$ dividi queat per G^2-F^2, pro variis valoribus ipsius n formulae inventae ita se habebunt:

si $n=1$, $\quad P^4 = \dfrac{(F^2G^2+1)P^2}{G^2} - \dfrac{F^2}{G^2}$ $\quad$ et $\quad Z = \dfrac{GP}{F}$,

si $n=2$, $\quad P^4 = \dfrac{2F^2(F^2G^2+1)P^2}{G^2(G^2+F^2)} - \dfrac{F^4}{G^4}$ $\quad$ et $\quad Z = \dfrac{G^2P}{F^2}$,

si $n=3$, $\quad P^4 = \dfrac{3F^4(F^2G^2+1)P^2}{G^2(G^4+F^2G^2+F^4)} - \dfrac{F^6}{G^6}$ $\quad$ et $\quad Z = \dfrac{G^3P}{F^3}$,

si $n=4$, $\quad P^4 = \dfrac{4F^6(F^2G^2+1)P^2}{G^2(G^6+F^2G^4+F^4G^2+F^6)} - \dfrac{F^8}{G^8}$ $\quad$ et $\quad Z = \dfrac{G^4P}{F^4}$,

etc. $\qquad$ etc. $\qquad$ etc.

COROLLARIUM 3

32. Ex solutione ceterum apparet pari modo pro arcu dato quocunque fg inveniri posse alium pz, qui illum arcum n vicibus sumtum data quantitate superet vel ab eo deficiat; ut enim sit Arc. $pz - n$ Arc. $fg = D$, resolvi oportebit hanc aequationem $\mathfrak{Z} - \mathfrak{P} = n(\mathfrak{G} - \mathfrak{F}) + D$, quae non habet plus difficultatis, quam si esset $D = 0$.

SCHOLION

33. Haec quidem, quae de circulo et parabola hic protuli, iam dudum satis sunt cognita, et quia utriusque rectificatio quasi in potestate est (quae enim vel a quadratura circuli vel a logarithmis pendent, in ordinem quantitatum algebraicarum propemodum recipiuntur), nulli omnino difficultati sunt subiecta; ea tamen nihilominus aliquanto uberius hic exponere visum est, quod ex methodo prorsus singulari consequuntur. Quod autem imprimis notatu dignum est, haec methodus ad comparationem aliarum quoque curvarum manuducit, quarum rectificatio per calculum solitum nullo modo expediri

potest; ita ut ex eodem quasi fonte plurimae eximiae affectiones tam cognitae quam incognitae hauriri queant, ex quo Analysi non contemnenda incrementa accedere censeri debebunt.

SECTIO SECUNDA
CONTINENS EVOLUTIONEM HUIUS AEQUATIONIS

$$0 = \alpha + \gamma(xx + yy) + 2\delta xy + \zeta xxyy$$

I

Extrahatur ex hac aequatione singillatim radix utriusque quantitatis variabilis x et y ac reperietur

$$y = \frac{-\delta x + \sqrt{(\delta\delta xx - (\alpha + \gamma xx)(\gamma + \zeta xx))}}{\gamma + \zeta xx},$$

$$x = \frac{-\delta y - \sqrt{(\delta\delta yy - (\alpha + \gamma yy)(\gamma + \zeta yy))}}{\gamma + \zeta yy}.$$

Ponatur brevitatis gratia

$$-\alpha\gamma = Ap, \quad \delta\delta - \gamma\gamma - \alpha\zeta = Cp \quad \text{et} \quad -\gamma\zeta = Ep$$

eritque

$$\gamma y + \delta x + \zeta xxy = \sqrt{p}(A + Cxx + Ex^4),$$

$$\gamma x + \delta y + \zeta xyy = -\sqrt{p}(A + Cyy + Ey^4).$$

II

Si igitur coefficientes A, C, E fuerint dati, ex iis litterarum graecarum valores facile definiuntur. Erit enim

$$\alpha = \frac{-Ap}{\gamma}, \quad \zeta = \frac{-Ep}{\gamma} \quad \text{et} \quad \delta = \sqrt{\left(\gamma\gamma + Cp + \frac{AEpp}{\gamma\gamma}\right)}.$$

Valores ergo γ et p arbitrio nostro relinquuntur atque alterum quidem sine ulla restrictione ad lubitum determinare licet. Ponatur ergo $\gamma\gamma = A$ et $p = cc$ fietque

$$\alpha = -cc\sqrt{A}, \quad \gamma = \sqrt{A}, \quad \delta = \sqrt{(A + Ccc + Ec^4)} \quad \text{et} \quad \zeta = \frac{-Ecc}{\sqrt{A}}$$

et aequatio canonica hanc induet formam

$$0 = -Acc + A(xx + yy) + 2xy\sqrt{A(A + Ccc + Ec^4)} - Eccxxyy.$$

III

Antequam autem his litteris maiusculis utamur, differentiemus ipsam aequationem propositam

$$dx(\gamma x + \delta y + \zeta xyy) + dy(\gamma y + \delta x + \zeta xxy) = 0,$$

quae abit in hanc

$$\frac{dx}{\gamma y + \delta x + \zeta xxy} = \frac{-dy}{\gamma x + \delta y + \zeta xyy}.$$

Substituendo ergo loco horum denominatorum valores surdos primo inventos habebimus per $\sqrt{p}$ multiplicando

$$\frac{dx}{\sqrt{(A + Cxx + Ex^4)}} = \frac{dy}{\sqrt{(A + Cyy + Ey^4)}}.$$

IV

Proposita ergo hac aequatione differentiali

$$\frac{dx}{\sqrt{(A + Cxx + Ex^4)}} = \frac{dy}{\sqrt{(A + Cyy + Ey^4)}}$$

eius aequatio integralis erit

$$0 = -Acc + A(xx + yy) + 2xy\sqrt{A(A + Ccc + Ec^4)} - Eccxxyy,$$

quae cum constantem novam c ab arbitrio nostro pendentem involvat, erit adeo integralis completa. Inde autem oritur

$$y = \frac{-x\sqrt{A(A + Ccc + Ec^4)} \pm c\sqrt{A(A + Cxx + Ex^4)}}{A - Eccxx},$$

ubi quidem signa radicalium pro lubitu mutare licet.

V

Cum igitur posita nostra aequatione canonica sit

$$\int \frac{dx}{\sqrt{(A + Cxx + Ex^4)}} - \int \frac{dy}{\sqrt{(A + Cyy + Ey^4)}} = \text{Const.},$$

ponamus ad alias integrationes eruendas

$$\int\frac{xxdx}{\sqrt{(A+Cxx+Ex^4)}}-\int\frac{yydy}{\sqrt{(A+Cyy+Ey^4)}}=V;$$

erit ergo loco radicalium valores praecedentes restituendo

$$\frac{xxdx}{\gamma y+\delta x+\zeta xxy}+\frac{yydy}{\gamma x+\delta y+\zeta xyy}=\frac{dV}{\sqrt{p}}$$

hincque porro

$$xxdx(\gamma x+\delta y+\zeta xyy)+yydy(\gamma y+\delta x+\zeta xxy)$$
$$=\frac{dV}{\sqrt{p}}\big(\gamma\delta(xx+yy)+(\gamma\gamma+\delta\delta)xy+\zeta\zeta x^3y^3+\gamma\zeta xy(xx+yy)+2\delta\zeta xxyy\big).$$

VI

Ponamus ad hanc aequationem concinniorem reddendam $xx+yy=tt$ et $xy=u$, ut sit

$$0=\alpha+\gamma tt+2\delta u+\zeta uu,$$

et aequatio nostra differentialis erit

$$\gamma(x^3dx+y^3dy)+\delta u(xdx+ydy)+\zeta uu(xdx+ydy)$$
$$=\frac{dV}{\sqrt{p}}\big(\gamma\delta tt+(\gamma\gamma+\delta\delta)u+\gamma\zeta ttu+2\delta\zeta uu+\zeta\zeta u^3\big).$$

At est

$$xdx+ydy=tdt$$

et ob $x^4+y^4=t^4-2uu$ erit

$$x^3dx+y^3dy=t^3dt-udu.$$

Porro aequatio canonica differentiata dat $\gamma tdt+\delta du+\zeta udu=0$ ideoque

$$tdt=\frac{-\delta du-\zeta udu}{\gamma},$$

unde fit

$$xdx+ydy=-\frac{\delta}{\gamma}du-\frac{\zeta}{\gamma}udu\quad\text{et}\quad x^3dx+y^3dy=-\frac{\delta}{\gamma}ttdu-\frac{\zeta}{\gamma}ttudu-udu.$$

VII

His igitur valoribus substitutis obtinebimus

$$du\left(-\delta tt-\zeta ttu-\gamma u-\frac{\delta\delta}{\gamma}u-\frac{2\delta\zeta}{\gamma}uu-\frac{\zeta\zeta}{\gamma}u^3\right)$$
$$=\frac{dV}{\sqrt{p}}\left(\gamma\delta tt+(\gamma\gamma+\delta\delta)u+\gamma\zeta ttu+2\delta\zeta uu+\zeta\zeta u^3\right),$$

quae sponte abit in

$$\frac{-du}{\gamma}=\frac{dV}{\sqrt{p}},$$

ita ut sit

$$V=\frac{-u\sqrt{p}}{\gamma}\quad\text{seu}\quad V=\frac{-xy\sqrt{p}}{\gamma}.$$

Facto ergo $p=cc$ erit

$$\int\frac{xxdx}{\sqrt{(A+Cxx+Ex^4)}}-\int\frac{yydy}{\sqrt{(A+Cyy+Ey^4)}}=\text{Const.}-\frac{cxy}{\sqrt{A}},$$

siquidem fuerit

$$0=-Acc+A(xx+yy)+2xy\sqrt{A(A+Ccc+Ec^4)}-Eccxxyy$$

seu

$$y=\frac{c\sqrt{A(A+Cxx+Ex^4)}-x\sqrt{A(A+Ccc+Ec^4)}}{A-Eccxx}.$$

VIII

Quo nunc rem generalius complectamur, ponamus

$$\int\frac{x^n dx}{\sqrt{(A+Cxx+Ex^4)}}-\int\frac{y^n dy}{\sqrt{(A+Cyy+Ey^4)}}=V;$$

erit

$$x^n dx(\gamma x+\delta y+\zeta xyy)+y^n dy(\gamma y+\delta x+\zeta xxy)$$
$$=\frac{dV}{\sqrt{p}}(\gamma\delta tt+(\gamma\gamma+\delta\delta)u+\gamma\zeta ttu+2\delta\zeta uu+\zeta\zeta u^3)$$

posito ut ante $xx+yy=tt$ et $xy=u$. Erit ergo $xx-yy=\sqrt{(t^4-4uu)}$, unde

$$x=\sqrt{\frac{tt+\sqrt{(t^4-4uu)}}{2}}\quad\text{et}\quad y=\sqrt{\frac{tt-\sqrt{(t^4-4uu)}}{2}}$$

seu

$$x=\frac{1}{2}\sqrt{(tt+2u)}+\frac{1}{2}\sqrt{(tt-2u)}\quad\text{et}\quad y=\frac{1}{2}\sqrt{(tt+2u)}-\frac{1}{2}\sqrt{(tt-2u)}.$$

Quare differentiando habebitur

$$dx = \frac{tdt+du}{2\sqrt{(tt+2u)}} + \frac{tdt-du}{2\sqrt{(tt-2u)}} = \frac{du(\gamma-\delta-\zeta u)}{2\gamma\sqrt{(tt+2u)}} - \frac{du(\gamma+\delta+\zeta u)}{2\gamma\sqrt{(tt-2u)}}.$$

IX

Porro vero est

$$\gamma x + \delta y + \zeta xyy = \left(\frac{1}{2}(\gamma+\delta) + \frac{1}{2}\zeta u\right)\sqrt{(tt+2u)} + \left(\frac{1}{2}(\gamma-\delta) - \frac{1}{2}\zeta u\right)\sqrt{(tt-2u)},$$

unde colligitur

$$\begin{aligned} &dx(\gamma x + \delta y + \zeta xyy) \\ &= \frac{du}{4\gamma}(\gamma+\delta+\zeta u)(\gamma-\delta-\zeta u) + \frac{du}{4\gamma}(\gamma-\delta-\zeta u)(\gamma-\delta-\zeta u)\sqrt{\frac{tt-2u}{tt+2u}} \\ &- \frac{du}{4\gamma}(\gamma-\delta-\zeta u)(\gamma+\delta+\zeta u) - \frac{du}{4\gamma}(\gamma+\delta+\zeta u)(\gamma+\delta+\zeta u)\sqrt{\frac{tt+2u}{tt-2u}} \end{aligned}$$

seu

$$dx(\gamma x + \delta y + \zeta xyy) = \frac{-du}{\gamma\sqrt{(t^4-4uu)}}(\gamma\delta tt + \gamma\zeta ttu + (\gamma\gamma+\delta\delta)u + 2\delta\zeta uu + \zeta\zeta u^3),$$

et quia

$$dy(\gamma y + \delta x + \zeta xxy) = -dx(\gamma x + \delta y + \zeta xyy),$$

erit

$$\frac{dV}{\sqrt{p}} = \frac{-du(x^n-y^n)}{\gamma\sqrt{(t^4-4uu)}} \quad \text{et} \quad V = \frac{-\sqrt{p}}{\gamma}\int\frac{(x^n-y^n)\,du}{\sqrt{(t^4-4uu)}}.$$

X

Ut haec formula evadat integrabilis, oportet pro n scribi numerum parem, ut etiam usus huius formae plerumque exigit. Quare si

$n=0$, erit	$x^0-y^0=0$	$V=\text{Const.}$
$n=2$,	$x^2-y^2=\sqrt{(t^4-4uu)}$	$V=\frac{-u\sqrt{p}}{\gamma}$
$n=4$,	$x^4-y^4=tt\sqrt{(t^4-4uu)}$	$V=\frac{-\sqrt{p}}{\gamma}\int ttdu$
$n=6$,	$x^6-y^6=(t^4-uu)\sqrt{(t^4-4uu)}$	$V=\frac{-\sqrt{p}}{\gamma}\int(t^4-uu)du$
$n=8$,	$x^8-y^8=(t^6-2ttuu)\sqrt{(t^4-4uu)}$	$V=\frac{-\sqrt{p}}{\gamma}\int(t^6-2ttuu)du$
etc.	etc.	etc.

XI

Cum vero sit $tt = \frac{-\alpha - 2\delta u - \zeta uu}{\gamma}$, erit

$$\int ttdu = \frac{-\alpha u}{\gamma} - \frac{\delta uu}{\gamma} - \frac{\zeta u^3}{3\gamma},$$

$$\int (t^4 - uu)du = \frac{\alpha\alpha}{\gamma\gamma}u + \frac{2\alpha\delta}{\gamma\gamma}uu + \frac{(4\delta\delta + 2\alpha\zeta - \gamma\gamma)}{3\gamma\gamma}u^3 + \frac{\delta\zeta}{\gamma\gamma}u^4 + \frac{\zeta\zeta}{5\gamma\gamma}u^5.$$

Ex his introductis litteris maiusculis A, C, E una cum constanti arbitraria c aequatio in fine art. VII data satisfaciet huic aequationi integrali

$$\int \frac{dx(\mathfrak{A} + \mathfrak{C}xx + \mathfrak{E}x^4)}{\sqrt{(A + Cxx + Ex^4)}} - \int \frac{dy(\mathfrak{A} + \mathfrak{C}yy + \mathfrak{E}y^4)}{\sqrt{(A + Cyy + Ey^4)}}$$
$$= \text{Const.} - \frac{\mathfrak{C}cxy}{\sqrt{A}} - \frac{\mathfrak{E}cxy}{\sqrt{A}}\left(cc - xy\sqrt{\frac{A + Ccc + Ec^4}{A}} + \frac{Eccxxyy}{3A}\right).$$

Unde sequentes curvarum comparationes adipiscimur.

COMPARATIO ARCUUM ELLIPSIS

1. Expressio simplicissima ad hoc genus pertinens est utique curva lemniscata, sed quia comparationem arcuum eius iam satis prolixe sum persecutus, hic statim ab ellipsi incipiam. Sit igitur ACB (Fig. 3) quadrans ellipticus, cuius alter semiaxis $CA = 1$, alter $CB = k$. Eritque posita abscissa quacunque $CP = z$ arcus ei respondens

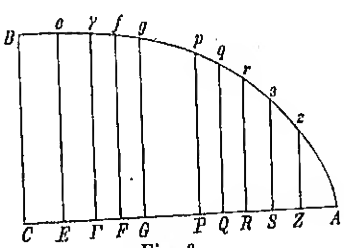

Fig. 3.

$$Bp = \int dz\sqrt{\frac{1 - (1 - kk)zz}{1 - zz}}.$$

Sit brevitatis gratia $1 - kk = n$, ita ut $\sqrt{n}$ denotet distantiam foci a centro C, hincque fiet

$$\text{Arc. } Bp = \int \frac{dz\sqrt{(1 - nzz)}}{\sqrt{(1 - zz)}}.$$

2. Reddatur formulae huius numerator rationalis, ut prodeat

$$\text{Arc. } Bp = \int \frac{dz(1 - nzz)}{\sqrt{(1 - (n + 1)zz + nz^4)}};$$

ad quam formam ut formulae superiores reducantur, poni oportet $A = 1$, $C = -n - 1$, $E = n$, $\mathfrak{A} = 1$, $\mathfrak{C} = -n$, $\mathfrak{E} = 0$; quo facto habebimus pro differentia duorum arcuum ellipticorum

$$\int dx \sqrt{\frac{1-nxx}{1-xx}} - \int dy \sqrt{\frac{1-nyy}{1-yy}} = \text{Const.} + ncxy,$$

siquidem abscissa y ex abscissa x ita determinetur, ut sit

$$y = \frac{c\sqrt{(1-xx)(1-nxx)} - x\sqrt{(1-cc)(1-ncc)}}{1-nccxx}$$

sive

$$0 = -cc + xx + yy + 2xy\sqrt{(1-cc)(1-ncc)} - nccxxyy.$$

3. Denotet $\Pi.z$ arcum ellipsis abscissae z respondentem ac nostra aequatio inventa hanc induet formam

$$\Pi.x - \Pi.y = \text{Const.} + ncxy,$$

posito autem $x = 0$ fit $y = c$, unde Const. $= -\Pi.c$. Ergo

$$\Pi.c + \Pi.x - \Pi.y = ncxy.$$

Sin autem sumto $\sqrt{(1-cc)(1-ncc)}$ negativo, ut sit

$$y = \frac{c\sqrt{(1-xx)(1-nxx)} + x\sqrt{(1-cc)(1-ncc)}}{1-nccxx}$$

fiet

$$\Pi.y - \Pi.c - \Pi.x = -ncxy \quad \text{sive} \quad \Pi.c - (\Pi.y - \Pi.x) = ncxy$$

ut ante.

4. Ternae autem quantitates c, x, y ita a se invicem pendent, ut habita signorum ratione inter se permutari possint; si enim ad abbreviandum ponatur

$$\sqrt{(1-cc)(1-ncc)} = C, \quad \sqrt{(1-xx)(1-nxx)} = X, \quad \sqrt{(1-yy)(1-nyy)} = Y,$$

erit

$$y = \frac{cX + xC}{1-nccxx}, \quad x = \frac{yC - cY}{1-nccyy}, \quad c = \frac{yX - xY}{1-nxxyy},$$

ex quibus per combinationem eliciuntur sequentes formulae

$$yy - xx = c(yX + xY), \qquad xX + yY = (nccxy + C)(yX + xY),$$
$$yy - cc = x(yC + cY), \qquad cC - xX = (ncxyy - Y)(xC - cX),$$
$$xx - cc = y(xC - cX), \qquad cC + yY = (ncxxy + X)(yC + cY)$$

ac denique

$$2xyC = xx + yy - cc - nccxxyy,$$
$$2cyX = cc + yy - xx - nccxxyy,$$
$$-2cxY = cc + xx - yy - nccxxyy.$$

PROBLEMA 1

5. *Dato arcu elliptico Be (Fig. 3, p. 325) in vertice B terminato abscindere a quovis puncto dato f alium arcum fg, ut eorum differentia $fg - Be$ geometrice assignari queat.*

SOLUTIO

Sint abscissae datae $CE = e$, $CF = f$ et quaesita $Cg = g$; erit

$$\text{Arc. } Be = \Pi.\, e \quad \text{et} \quad \text{Arc. } fg = \Pi.\, g - \Pi.\, f;$$

ut igitur arcuum fg et Be differentia fiat geometrica, necesse est, ut sit $\Pi. e - (\Pi. g - \Pi. f) =$ quantitati algebraicae. Hoc autem, ut vidimus, evenit, si

$$g = \frac{e\sqrt{(1-ff)(1-nff)} + f\sqrt{(1-ee)(1-nee)}}{1 - neeff}.$$

Quodsi ergo abscissae $CG = g$ hic tribuatur valor, erit

$$\text{Arc. } Be - \text{Arc. } fg = nefg,$$

posito scilicet $CA = 1$ et $CB = k$ atque $n = 1 - kk$. Q. E. I.

COROLLARIUM 1

6. Poterit etiam a puncto dato f versus B accedendo eiusmodi arcus $f\gamma$ abscindi, ut differentia $Be - f\gamma$ fiat algebraica. Posita enim abscissa $C\Gamma = \gamma$

capiatur

$$\gamma = \frac{f\sqrt{(1-ee)(1-nee)} - e\sqrt{(1-ff)(1-nff)}}{1-neeff}$$

eritque

$$\text{Arc. } Be - \text{Arc. } f\gamma = nef\gamma.$$

COROLLARIUM 2

7. Erit ergo quoque arcuum $f\gamma$ et fg differentia geometrice assignabilis; habebitur enim

$$\text{Arc. } f\gamma - \text{Arc. } fg = nef(g-\gamma).$$

Est autem

$$g - \gamma = \frac{2e\sqrt{(1-ff)(1-nff)}}{1-neeff},$$

sive cum sit

$$2fg\sqrt{(1-ee)(1-nee)} = ff + gg - ee - neeffgg$$

et

$$2f\gamma\sqrt{(1-ee)(1-nee)} = ff + \gamma\gamma - ee - neeff\gamma\gamma,$$

erit

$$ee = \frac{ff - \gamma g}{1 - nff\gamma g}$$

et

$$g - \gamma = 2\sqrt{(1-ff)(1-nff)(ff-\gamma g)(1-nff\gamma g)}$$

atque

$$\text{Arc. } f\gamma - \text{Arc. } fg = 2nf(ff-\gamma g)\sqrt{(1-ff)(1-nff)}.$$

COROLLARIUM 3

8. Cum sit

$$g = \frac{e\sqrt{(1-ff)(1-nff)} + f\sqrt{(1-ee)(1-nee)}}{1-neeff},$$

erit

$$\sqrt{(1-gg)} = \frac{\sqrt{(1-ee)(1-ff)} - ef\sqrt{(1-nee)(1-nff)}}{1-neeff}$$

et

$$\sqrt{(1-ngg)} = \frac{\sqrt{(1-nee)(1-nff)} - nef\sqrt{(1-ee)(1-ff)}}{1-neeff}$$

hincque

$$\frac{g}{\sqrt{(1-gg)}}=\frac{e\sqrt{(1-ee)(1-nff)}+f\sqrt{(1-ff)(1-nee)}}{1-ee-ff+neeff},$$

$$\frac{\sqrt{(1-ngg)}}{\sqrt{(1-gg)}}=\frac{\sqrt{(1-ee)(1-nee)(1-ff)(1-nff)}+(1-n)ef}{1-ee-ff+neeff},$$

$$\frac{g\sqrt{(1-ngg)}}{\sqrt{(1-gg)}}=\frac{e(1-2nff+nf^4)\sqrt{(1-ee)(1-nee)}+f(1-2nee+ne^4)\sqrt{(1-ff)(1-nff)}}{(1-ee-ff+neeff)(1-neeff)},$$

$$\sqrt{(1-gg)(1-ngg)}$$

$$=\frac{ef(2n(ee+ff)-(n+1)(1+neeff))+(1+neeff)\sqrt{(1-ee)(1-nee)(1-ff)(1-nff)}}{(1-neeff)^2}.$$

Huiusmodi autem formulae inveniuntur, si simpliciores inverso quoque exprimantur; sic erit

$$\frac{1}{g}=\frac{f\sqrt{(1-ee)(1-nee)}-e\sqrt{(1-ff)(1-nff)}}{ff-ee},$$

$$\frac{1}{\sqrt{(1-gg)}}=\frac{\sqrt{(1-ee)(1-ff)}+ef\sqrt{(1-nee)(1-nff)}}{1-ee-ff+neeff},$$

$$\frac{1}{\sqrt{(1-ngg)}}=\frac{\sqrt{(1-nee)(1-nff)}+nef\sqrt{(1-ee)(1-ff)}}{1-nee-nff+neeff}.$$

COROLLARIUM 4

9. Has formulas ideo evolvere visum est, ut, si fieri posset, ex iis eiusmodi relatio inter e, f, g determinaretur, ut functio quaepiam ipsius g fieret aequalis producto ex functionibus similibus ipsarum e et f. Verum huiusmodi expressio, qualis pro parabola est reperta, hic pro ellipsi non tam facile erui posse videtur. Simpliciores autem harum formularum combinationes dant

$$\sqrt{(1-gg)}+ef\sqrt{(1-ngg)}=\sqrt{(1-ee)(1-ff)},$$

$$\sqrt{(1-ngg)}+nef\sqrt{(1-gg)}=\sqrt{(1-nee)(1-nff)}.$$

COROLLARIUM 5

10. Ut igitur sit Arc. Be — Arc. $fg = nefg$, relatio inter abscissas e, f, g ita debet esse comparata, ut sit vel

$$g = \frac{e\sqrt{(1-ff)(1-nff)}+f\sqrt{(1-ee)(1-nee)}}{1-neeff}$$

vel

$$f = \frac{g\sqrt{(1-ee)(1-nee)}-e\sqrt{(1-gg)(1-ngg)}}{1-neegg}$$

vel

$$e = \frac{g\sqrt{(1-ff)(1-nff)}-f\sqrt{(1-gg)(1-ngg)}}{1-nffgg}.$$

COROLLARIUM 6

11. Si punctum g statuatur in vertice A, erit $g=1$ et $f=\sqrt{\frac{1-ee}{1-nee}}$, qui est casus a Com. FAGNANO datus. Nunc igitur hoc problema de duobus arcubus ellipseos, quorum differentia sit geometrice assignabilis, multo generalius est solutum, cum dato arcu Be alter terminus arcus quaesiti, ubi libuerit, accipi queat.

COROLLARIUM 7

12. Effici autem omnino nequit, ut horum arcuum differentia evanescat, ita ut duo arcus dissimiles ellipsis inter se aequales exhiberi queant; ut enim hoc eveniret, vel e vel f vel g evanescere deberet, unde vel arcus evanescentes vel similes prodituri essent.

PROBLEMA 2

13. *Dato ellipsis arcu quocunque fg* (Fig. 3, p. 325) *a puncto quovis dato p alium arcum pq abscindere, ita ut horum duorum arcuum differentia sit geometrice assignabilis.*

SOLUTIO

Positis abscissis pro arcu dato $CF=f$, $CG=g$ et pro quaesito $CP=p$ et $CQ=q$, quarum quidem altera, vel p vel q, pro lubitu assumi poterit. In subsidium nunc vocetur arcus Be abscissae $CE=e$ respondens, qui per problema 1 ita sit comparatus, ut fiat

$$\text{Arc. } Be - \text{Arc. } fg = nefg \quad \text{et} \quad \text{Arc. } Be - \text{Arc. } pq = nepq.$$

Hoc autem ut eveniat, necesse est, ut sit

$$e = \frac{g\sqrt{(1-ff)(1-nff)} - f\sqrt{(1-gg)(1-ngg)}}{1-nffgg}$$

pariterque

$$e = \frac{q\sqrt{(1-pp)(1-npp)} - p\sqrt{(1-qq)(1-nqq)}}{1-nppqq}.$$

His igitur duobus valoribus inter se aequatis determinabitur q per f, g et p, uti problema exigit; et quia abscissa e est cognita, erit

$$\text{Arc. } fg - \text{Arc. } pq = ne(pq - fg).$$

Sicque differentia arcuum fg et pq est geometrica et arcus quaesiti pq alter terminus ab arbitrio nostro pendet. Q. E. I.

COROLLARIUM 1

14. Datis ergo punctis f, g et p quartum punctum q seu eius abscissa $CQ = q$ ex hac aequatione debet definiri

$$\frac{g\sqrt{(1-ff)(1-nff)} - f\sqrt{(1-gg)(1-ngg)}}{1-nffgg} = \frac{q\sqrt{(1-pp)(1-npp)} - p\sqrt{(1-qq)(1-nqq)}}{1-nppqq},$$

vel quia haec formula non parum est complicata, quantitas e ex huiusmodi aequationibus simplicioribus eliminari poterit

$$\sqrt{(1-ee)} - fg\sqrt{(1-nee)} = \sqrt{(1-ff)(1-gg)}$$

et

$$\sqrt{(1-ee)} - pq\sqrt{(1-nee)} = \sqrt{(1-pp)(1-qq)},$$

$$\sqrt{(1-nee)} - nfg\sqrt{(1-ee)} = \sqrt{(1-nff)(1-ngg)}$$

et

$$\sqrt{(1-nee)} - npq\sqrt{(1-ee)} = \sqrt{(1-npp)(1-nqq)};$$

unde elicitur

$$\sqrt{(1-ff)(1-gg)} - pq\sqrt{(1-nff)(1-ngg)}$$
$$= \sqrt{(1-pp)(1-qq)} - fg\sqrt{(1-npp)(1-nqq)}$$

vel etiam

$$\sqrt{(1-nff)(1-ngg)} - npq\sqrt{(1-ff)(1-gg)}$$
$$= \sqrt{(1-npp)(1-nqq)} - nfg\sqrt{(1-pp)(1-qq)}.$$

COROLLARIUM 2

15. Ut ambo hi arcus fg et pq fiant inter se aequales, oportet sit $pq = fg$. Ponatur $pp + qq = t$ et ambae postremae aequationes dabunt

$$\begin{aligned} &\sqrt{(1-ff)(1-gg)} - fg\sqrt{(1-nff)(1-ngg)} \\ &= \sqrt{(1-t+ffgg)} - fg\sqrt{(1-nt+nnffgg)}, \\ &\sqrt{(1-nff)(1-ngg)} - nfg\sqrt{(1-ff)(1-gg)} \\ &= \sqrt{(1-nt+nnffgg)} - nfg\sqrt{(1-t+ffgg)}, \end{aligned}$$

quarum haec per fg multiplicata ad illam addatur, ut prodeat

$$(1-nffgg)\sqrt{(1-ff)(1-gg)} = (1-nffgg)\sqrt{(1-t+ffgg)}$$

seu

$$1 - ff - gg + ffgg = 1 - t + ffgg$$

ideoque

$$t = ff + gg = pp + qq.$$

Unde sequitur arcum pq similem et aequalem futurum esse arcui fg.

PROBLEMA 3

16. *Dato arcu ellipsis quocunque fg* (Fig. 3, p. 325) *abscindere a dato puncto p alium arcum pqr, qui deficiat a duplo illius arcus fg quantitate algebraica, seu ut sit* 2 Arc. fg — Arc. pqr = *lineae rectae.*

SOLUTIO

Sint abscissae ut ante $CE = e$, $CF = f$, $CG = g$, $CP = p$, $CQ = q$ et $CR = r$, ubi Be est arcus a vertice B abscissus ab arcu fg dato geometrice discrepans; a quo etiam arcus pq et qr discrepent quantitatibus geometrice assignabilibus. Erit ergo

$$\text{I. } e = \frac{g\sqrt{(1-ff)(1-nff)} - f\sqrt{(1-gg)(1-ngg)}}{1-nffgg}$$

$$\text{II. } e = \frac{q\sqrt{(1-pp)(1-npp)} - p\sqrt{(1-qq)(1-nqq)}}{1-nppqq}$$

$$\text{III. } e = \frac{r\sqrt{(1-qq)(1-nqq)} - q\sqrt{(1-rr)(1-nrr)}}{1-nqqrr}.$$

Hinc si primum definiatur abscissa e ex eaque porro abscissae q et r, erit

$$\text{Arc. } fg - \text{Arc. } pq = ne(pq - fg),$$

$$\text{Arc. } fg - \text{Arc. } qr = ne(qr - fg),$$

quibus aequationibus additis habebitur

$$2 \text{ Arc. } fg - \text{Arc. } pqr = ne(pq + qr - 2fg).$$

Q. E. I.

COROLLARIUM 1

17. Quoniam dato arcu fg etiam arcus Be datur, spectemus e tanquam quantitatem cognitam eritque

$$p = \frac{q\sqrt{(1-ee)(1-nee)} - e\sqrt{(1-qq)(1-nqq)}}{1-neeqq},$$

$$r = \frac{q\sqrt{(1-ee)(1-nee)} + e\sqrt{(1-qq)(1-nqq)}}{1-neeqq},$$

unde fit

$$p + r = \frac{2q\sqrt{(1-ee)(1-nee)}}{1-neeqq}.$$

COROLLARIUM 2

18. Differentia ergo arcuum $2fg$ et pqr hoc modo determinatorum erit

$$2 \text{ Arc. } fg - \text{Arc. } pqr = 2ne\left(\frac{qq\sqrt{(1-ee)(1-nee)}}{1-neeqq} - fg\right).$$

Ut ergo arcus pqr exacte aequalis fiat duplo arcus fg, oportet esse

$$fg = \frac{qq\sqrt{(1-ee)(1-nee)}}{1-neeqq},$$

unde definitur

$$qq = \frac{fg}{neefg + \sqrt{(1-ee)(1-nee)}},$$

hincque porro inveniuntur p et r.

COROLLARIUM 3

19. Relatio autem abscissarum e, f, g hac aequatione exprimitur

$$ff + gg = ee + neeffgg + 2fg\sqrt{(1-ee)(1-nee)};$$

unde facillime duo arcus in ellipsi, quorum alter alterius sit duplus, hoc modo determinabuntur. Sumta pro lubitu abscissa e capiatur quoque pro lubitu valor producti fg; exhinc reperietur summa quadratorum $ff+gg$, unde utraque abscissa f et g seorsim reperietur. Inde vero porro colligitur abscissa q ex eaque denique abscissae p et r, ut arcus pqr fiat duplus arcus fg.

COROLLARIUM 4

20. Nihilo tamen minus arcus fg pro arbitrio assumi potest nec non alter terminus arcus quaesiti vel p vel r, ex quo deinceps definiri poterit alter terminus, ut arcus pqr fiat duplo maior quam arcus fg. Sed haec operatio multo fit molestior et calculum requirit prolixiorem.

COROLLARIUM 5

21. Si priore operatione utamur, qua quantitatibus e et fg arbitrarios valores tribuimus, cavendum est, ne inde valor ipsius q prodeat unitate maior seu $CQ > CA$; sic enim perveniretur ad imaginaria. Ut autem prodeat $q < 1$, capi debet $fg < \sqrt{\frac{1-ee}{1-nee}}$; at si capiatur $fg = \sqrt{\frac{1-ee}{1-nee}}$, fit

$$g = 1, \quad f = \sqrt{\frac{1-ee}{1-nee}} \quad \text{et} \quad q = 1$$

hincque

$$p + r = 2\sqrt{\frac{1-ee}{1-nee}} \quad \text{et} \quad p = r = \sqrt{\frac{1-ee}{1-nee}}.$$

Hoc ergo casu arcus fg in A terminatur et arcus pqr utrinque circa A aequaliter protenditur, uti est obvium.

EXEMPLUM

22. Ponamus $n = \frac{1}{2}$ et $ee = \frac{1}{2}$, ut semiaxis coniugatus ellipsis prodeat $CB = \sqrt{\frac{1}{2}}$ altero existente $CA = 1$. Quia nunc esse debet $fg < \sqrt{\frac{2}{3}}$, statuatur

$g = \frac{6}{7}\sqrt{\frac{2}{3}} = \frac{2\sqrt{6}}{7}$ ac reperietur

$$f = \frac{1}{\sqrt{2}}, \quad g = \frac{4\sqrt{3}}{7}, \quad \text{tum vero} \quad q = \frac{2\sqrt{2}}{3};$$

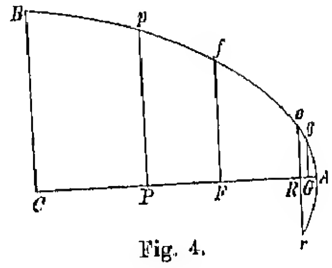

Fig. 4.

porro autem elicitur

$$p + r = \frac{6\sqrt{3}}{7} \quad \text{et} \quad r - p = \frac{\sqrt{10}}{7},$$

unde fit

$$p = \frac{6\sqrt{3} - \sqrt{10}}{14} \quad \text{et} \quad r = \frac{6\sqrt{3} + \sqrt{10}}{14}.$$

Hic casus fig. 4 repraesentatur, ubi arcus fg terminus g fere in verticem A cadit, punctum p vero ultra f versus B reperitur, at punctum r capi debet in ellipsis parte inferiori; ita ut arcus $pfgAr$ alterum arcum fg, cuius ille est duplus, totum in se complectatur.

SCHOLION

23. Si libuerit alia huiusmodi exempla expedire, in quibus radicalia non inter se implicentur, casus prodibunt simplicissimi ponendo $f = e$, unde prodit

$$g = \frac{2e}{1 - ne^4}\sqrt{(1 - ee)(1 - nee)};$$

tum vero reperitur

$$qq = \frac{2ee}{1 + ne^4},$$

ita ut esse oporteat

$$2ee < 1 + ne^4 \quad \text{seu} \quad ee > \frac{1 - \sqrt{(1 - n)}}{n},$$

alioquin loca p, q, r fuerint imaginaria. Hinc itaque pro terminis arcus quaesiti pqr elicitur

$$r + p = \frac{2e}{1 - ne^4}\sqrt{2(1 - ee)(1 - nee)(1 + ne^4)},$$

$$r - p = \frac{2e}{1 - ne^4}\sqrt{(1 - 2ee + ne^4)(1 - 2nee + ne^4)}$$

eritque, ut desideratur, Arc. $pqr = 2$ Arc. fg. Si ponamus semiaxem coniugatum

$CB = k = \frac{2(1-ee)}{1-2ee}$, ut sit

$$n = 1 - kk = \frac{-3+4ee}{(1-2ee)^2},$$

pleraeque irrationalitates evanescunt; fiet enim

$$f = e, \quad g = \frac{2e(1-2ee)}{1-3ee+4e^4}, \quad gg = \frac{2ee(1-2ee)^2}{1-4ee+e^4+4e^6}$$

atque

$$r + p = \frac{2e\sqrt{(2-8ee+2e^4+8e^6)}}{1-3ee+4e^4},$$

$$r - p = \frac{2e(1-ee)\sqrt{(1-16e^4)}}{1-3ee+4e^4}.$$

Debet ergo sumi $4ee < 1$, ne loca p et r fiant imaginaria. Imprimis autem notari meretur casus, quem in problemate sequente evolvam.

PROBLEMA 4

24. *In quadrante elliptico ACB* (Fig. 4, p. 335) *abscindere arcum fg, qui sit semissis totius arcus quadrantis $BfgA$.*

SOLUTIO

Cum arcus fg duplum esse debeat ipse quadrans BA, quantitates problematis ita debent definiri, ut punctum p in B et punctum r in A cadat. Erit ergo $p = 0$ et $r = 1$, unde fit

$$e = q \quad \text{et} \quad e = \sqrt{\frac{1-qq}{1-nqq}} = \sqrt{\frac{1-ee}{1-nee}}$$

seu

$$1 - 2ee + ne^4 = 0 \quad \text{ideoque} \quad ee = \frac{1-\sqrt{(1-n)}}{n}.$$

Cum autem posito $CB = k$ sit $n = 1 - kk$, erit

$$ee = \frac{1-k}{1-kk} = \frac{1}{1+k}$$

sicque habebimus

$$e = q = \frac{1}{\sqrt{(1+k)}}.$$

Tum vero, quia esse oportet $2fg = pq + qr$, erit

$$2fg = e = \frac{1}{\sqrt{(1+k)}}$$

atque

$$ff + gg = ee + \frac{1}{4} ne^4 + e\sqrt{(1-ee)(1-nee)}$$

sive

$$ff + gg = \frac{5+3k}{4+4k};$$

ergo ob $2fg = \frac{4\sqrt{(1+k)}}{4+4k}$ fiet

$$(f+g)^2 = \frac{5+3k+4\sqrt{(1+k)}}{4+4k} \quad \text{et} \quad (g-f)^2 = \frac{5+3k-4\sqrt{(1+k)}}{4+4k},$$

ergo

$$f = \sqrt{\frac{5+3k-\sqrt{(9+14k+9kk)}}{8+8k}} \quad \text{et} \quad g = \sqrt{\frac{5+3k+\sqrt{(9+14k+9kk)}}{8+8k}}$$

sicque puncta f et g determinantur, ut arcus fg sit semissis quadrantis AB.
Q. E. I.

COROLLARIUM 1

25. Quo hae formulae simpliciores evadant, ponatur semiaxis coniugatus

$$CB = k = \frac{1-4m}{1+4m} \quad \text{seu} \quad 4m = \frac{1-k}{1+k}$$

eritque

$$f = CF = \sqrt{\frac{1+m-\sqrt{(mm+\frac{1}{2})}}{2}} \quad \text{et} \quad g = CG = \sqrt{\frac{1+m+\sqrt{(mm+\frac{1}{2})}}{2}}.$$

COROLLARIUM 2

26. Vel in subsidium vocetur angulus quidam φ, cuius sinus sit

$$= \frac{\sqrt{(2m+\frac{1}{2})}}{m+1} \quad \text{seu} \quad \sin.\varphi = \frac{4\sqrt{(1+k)}}{5+3k},$$

eritque

$$CF = f = \sin.\tfrac{1}{2}\varphi\cdot\sqrt{\frac{5+3k}{4+4k}} \quad \text{et} \quad CG = g = \cos.\tfrac{1}{2}\varphi\cdot\sqrt{\frac{5+3k}{4+4k}}.$$

COROLLARIUM 3

27. Si sit $k=1$, quo casu ellipsis abit in circulum, erit $\sin.\varphi=\sqrt{\frac{1}{2}}$ ideoque $\varphi=45^{\circ}$ et ob $\sqrt{\frac{5+3k}{4+4k}}=1$ erit

$$CF=f=\sin.22\tfrac{1}{2}^{\circ} \quad \text{et} \quad CG=g=\cos.22\tfrac{1}{2}^{\circ}=\sin.67\tfrac{1}{2}^{\circ},$$

ita ut arcus fg prodeat 45°, qui utique est semissis quadrantis.

COROLLARIUM 4

28. Si ellipsis semiaxis coniugatus $CB=k$ evanescat prae $CA=1$, fiet $f=\frac{1}{2}$ et $g=1$; sin autem $CB=k$ sit quasi infinitus respectu $CA=1$, erit $f=0$ et $g=\sqrt{\frac{3}{4}}$, unde applicatae $Ff=k$ et $Gg=\frac{1}{2}k$, ita ut hi duo casus eodem recidant; utroque enim ellipsis confunditur cum linea recta.

COROLLARIUM 5

29. Si fuerit $k=\frac{5}{7}$, prodit $f=\sqrt{\frac{1}{6}}$ et $g=\sqrt{\frac{7}{8}}$. At si generalius ponatur $m=\frac{1-2uu}{4u}$, ut sit $k=\frac{2uu+u-1}{1+u-2uu}$, fiet $f=\sqrt{\frac{1-u}{2}}$ et $g=\sqrt{\frac{1+2u}{4u}}$. Iam ut utraque expressio fiat rationalis, sit $u=1-2ff$ fietque

$$k=\frac{1-5ff+4f^4}{3ff-4f^4} \quad \text{et} \quad g=\frac{\sqrt{(3-10ff+8f^4)}}{2(1-2ff)}.$$

Ergo f ita debet determinari, ut $3-10ff+8f^4$ fiat quadratum; quod cum eveniat casu $f=1$, ponatur $f=\frac{1-z}{1+z}$ eritque

$$3-10ff+8f^4=\frac{1-20z+86zz-20z^3+z^4}{(1+z)^4}.$$

Cuius numerator ergo quadratum effici debet, ita tamen, ut prodeat $f<1$ seu z affirmativum et unitate minus. Statim quidem apparet quadratum prodire posito $z=-\frac{3}{10}$; quia vero hic valor est negativus, ponatur $z=\frac{y-3}{10}$ eritque numerator ille

$$1-20z+86zz-20z^3+z^4=\frac{y^4-212y^3+10454yy-77108y+391\cdot 391}{10000}.$$

Posita huius radice $= \frac{yy-106y+391}{100}$ fit

$$y=\frac{1446}{391} \quad \text{et} \quad z=\frac{273}{3910}, \quad f=\frac{3637}{4183} \quad \text{et} \quad g=\frac{yy-106y+391}{200(1-2ff)(1+z)^2}$$

seu

$$g=\frac{yy-106y+391}{200(6z-1-zz)}=\frac{100zz-1000z+82}{200(6z-1-zz)}=\frac{647}{5986}.$$

Sicque casus exhiberi potest, in quo tam semiaxes ellipsis quam ambae abscissae f et g numeris rationalibus exprimuntur.

SCHOLION

30. Simili etiam modo si detur arcus ellipsis quicunque fg (Fig. 3, p. 325), a puncto quovis dato p alius assignari poterit arcus pz, qui datum multiplum arcus fg, puta $m\cdot fg$, superet quantitate algebraica; si enim abscissae ponantur $CF=f$, $CG=g$, $CP=p$, $CQ=q$, $CR=r$, $CS=s$, $CT=t$ et ab abscissa CP numerando fuerit $CZ=z$ ultima indici m respondens, tum in subsidium vocando arcum Be, cuius abscissa $Ce=e$, ut sit

$$e=\frac{g\sqrt{(1-ff)(1-nff)}-f\sqrt{(1-gg)(1-ngg)}}{1-nffgg},$$

ex data abscissa p sequentes ita determinentur

$$q=\frac{p\sqrt{(1-ee)(1-nee)}+e\sqrt{(1-pp)(1-npp)}}{1-neepp},$$

$$r=\frac{q\sqrt{(1-ee)(1-nee)}+e\sqrt{(1-qq)(1-nqq)}}{1-neeqq},$$

$$s=\frac{r\sqrt{(1-ee)(1-nee)}+e\sqrt{(1-rr)(1-nrr)}}{1-neerr},$$

etc.,

donec perveniatur ad ultimam z, quae a p numerando locum tenet indice m notatum. Quo facto erit

$$m.\,\text{Arc.}\,fg-\text{Arc.}\,pz=ne(pq+qr+rs+\cdots+yz-mfg).$$

Hinc igitur quoque punctum p ita definiri poterit, ut haec quantitas algebraica evanescat seu fiat

$$pq+qr+rs+\cdots+yz=mfg,$$

quo casu arcus pz exacte erit aequalis arcui fg toties sumto, quot numerus m continet unitates, seu erit Arc. $pz = m.$ Arc. fg. Dato ergo ellipsis arcu quocunque fg alius assignari poterit pz, qui ad illum datam teneat rationem, puta $m:1$. Quin etiam m poterit esse numerus fractus seu ista ratio ut numerus ad numerum $\mu:\nu$; nam quaeratur primo arcus pz, ut sit $pz = \mu . fg$, tum quaeratur alius $\pi\omega$, ut sit $\pi\omega = \nu . fg$, eritque $pz : \pi\omega = \mu : \nu$. Verum quo longius hic progrediamur, hae formulae continuo magis fiunt complicatae, ut calculum in genere expedire non liceat.

PROBLEMA 5

31. *In dato ellipseos quadrante AB* (Fig. 3, p. 325) *arcum abscindere fg, qui sit tertia pars totius quadrantis AB.*

SOLUTIO

Cum in genere fuerit determinatus arcus $pqrs$, qui sit triplus arcus fg, dum hic arcus tanquam cognitus est spectatus, nunc vicissim calculus ita instruatur, ut punctum p in B et punctum s in A incidat, seu ut sit $p = 0$ et $s = 1$. Formulae ergo modo exhibitae abibunt in has

$$q = e, \quad r = \frac{2e\sqrt{(1-ee)(1-nee)}}{1-ne^4} \quad \text{et} \quad 1 = \frac{r\sqrt{(1-ee)(1-nee)} + e\sqrt{(1-rr)(1-nrr)}}{1-neerr}$$

seu

$$r = \sqrt{\frac{1-ee}{1-nee}}$$

ob

$$r = \frac{s\sqrt{(1-ee)(1-nee)} - e\sqrt{(1-ss)(1-nss)}}{1-neess},$$

unde fit $2e(1-nee) = 1-ne^4$ seu

$$1 - 2e + 2ne^3 - ne^4 = 0$$

existente semiaxe $CA = 1$, $CB = k$ et $n = 1 - kk$. Primum ergo ex hac aequatione biquadratica definiri debet valor ipsius e, quae resolutio commode ita succedit. Sit $e = \frac{1}{x}$, ut habeatur $x^4 - 2x^3 + 2nx - n = 0$, ac ponatur ad secundum terminum tollendum $x = y + \frac{1}{2}$; prodibit

$$y^4 - \frac{3}{2}yy + (2n-1)y - \frac{3}{16} = 0,$$

uius factores fingantur $yy+\alpha y+\beta$ et $yy-\alpha y+\gamma$, eritque

$$\beta+\gamma=\alpha\alpha-\frac{3}{2},\quad \gamma-\beta=\frac{2n-1}{\alpha}\quad \text{et}\quad \beta\gamma=-\frac{3}{16},$$

unde elicimus

$$(\beta+\gamma)^2-(\gamma-\beta)^2=\alpha^4-3\alpha^2+\frac{9}{4}-\frac{(2n-1)^2}{\alpha\alpha}=4\beta\gamma=-\frac{3}{4}$$

ideoque

$$\alpha^6-3\alpha^4+3\alpha^2=(2n-1)^2;$$

subtrahatur utrinque 1, ut cubus fiat completus

$$(\alpha\alpha-1)^3=4nn-4n,$$

ergo

$$\alpha\alpha=1+\sqrt[3]{4n(n-1)}=1-\sqrt[3]{4nkk}\quad \text{et}\quad \alpha=\sqrt{(1-\sqrt[3]{4nkk})}.$$

Invento ergo α erit

$$\beta=\frac{1}{2}\alpha\alpha-\frac{3}{4}-\frac{2n-1}{2\alpha}\quad \text{et}\quad \gamma=\frac{1}{2}\alpha\alpha-\frac{3}{4}+\frac{2n-1}{2\alpha}$$

indeque

$$y=-\frac{1}{2}\alpha\pm\sqrt{\left(\frac{3}{4}-\frac{1}{4}\alpha\alpha\pm\frac{2n-1}{2\alpha}\right)}=\frac{-\alpha\alpha\pm\sqrt{(3\alpha\alpha-\alpha^4\pm2(2n-1)\alpha)}}{2\alpha},$$

unde obtinetur

$$e=\frac{2}{2y+1}.$$

Porro debet esse $3fg=pq+qr+rs$ seu

$$3fg=(1+e)\sqrt{\frac{1-ee}{1-nee}}\quad \text{ideoque}\quad fg=\frac{1}{3}(1+e)\sqrt{\frac{1-ee}{1-nee}},$$

ex quo obtinemus

$$ff+gg=ee+\frac{1}{9}nee(1+e)^2\cdot\frac{1-ee}{1-nee}+\frac{2}{3}(1+e)(1-ee).$$

Cognitis igitur valoribus fg et $ff+gg$ seorsim abscissae $CF=f$ et $CG=g$ reperientur, quae arcum determinabunt fg praecise subtriplum totius quadrantis AB. Q. E. I.

COMPARATIO ARCUUM HYPERBOLAE

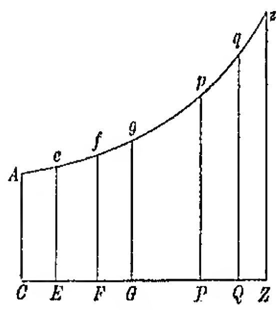

Fig. 5.

32. Sit C (Fig. 5) centrum hyperbolae, cuius semiaxis transversus $CA = k$ et semiaxis coniugatus $= 1$. Hinc sumta super axe coniugato a centro C abscissa quacunque $CZ = z$ erit applicata $Zz = k\sqrt{(1+zz)}$, unde arcus

$$Az = \int dz \sqrt{\frac{1+(1+kk)zz}{1+zz}}$$

$$= \int \frac{dz(1+(1+kk)zz)}{\sqrt{(1+(2+kk)zz+(1+kk)z^4)}}.$$

33. Ponatur brevitatis gratia $1+kk = n$, ita ut n sit numerus affirmativus unitate maior, eritque arcus hyperbolae quicunque

$$Az = \int \frac{dz(1+nzz)}{\sqrt{(1+(n+1)zz+nz^4)}}.$$

Poni igitur in n° XI oportet $A=1$, $C=n+1$, $E=n$, $\mathfrak{A}=1$, $\mathfrak{C}=n$ et $\mathfrak{E}=0$. Unde, si fuerit

$$y = \frac{c\sqrt{(1+xx)(1+nxx)} - x\sqrt{(1+cc)(1+ncc)}}{1-nccxx},$$

habebimus

$$\int dx \sqrt{\frac{1+nxx}{1+xx}} - \int dy \sqrt{\frac{1+nyy}{1+yy}} = \text{Const.} - ncxy.$$

34. Denotet $\Pi.x$ arcum abscissae x respondentem et $\Pi.y$ arcum abscissae y respondentem. Quia facto $x=0$ fit $y=c$, erit

$$\Pi.x - \Pi.y = -\Pi.c - ncxy \quad \text{seu} \quad \Pi.y - \Pi.x - \Pi.c = ncxy.$$

35. Ob $\sqrt{(1+cc)(1+ncc)}$ ambiguum poni quoque poterit

$$y = \frac{c\sqrt{(1+xx)(1+nxx)} + x\sqrt{(1+cc)(1+ncc)}}{1-nccxx}$$

eritque $\Pi.y - \Pi.x - \Pi.c = ncxy$ secundum ea, quae de ellipsi § 3 sunt exposita; atque hinc sequens problema solvi poterit.

PROBLEMA 6

36. *Dato arcu hyperbolae Ae (Fig. 5, p. 342) a vertice sumto abscindere a novis dato puncto f alium arcum fg, ut differentia horum arcuum fg et Ae sit geometrice assignabilis.*

SOLUTIO

Ponatur arcus propositi Ae abscissa $CE = e$, abscissa data $CF = f$ et quaesita $CG = g$; statuatur porro

$$g = \frac{e\sqrt{(1+ff)(1+nff)} + f\sqrt{(1+ee)(1+nee)}}{1-neeff}$$

eritque $\Pi.g - \Pi.f - \Pi.e = nefg$. At est

$$\Pi.g - \Pi.f = \text{Arc.}\, fg \quad \text{et} \quad \Pi.e = \text{Arc.}\, Ae,$$

unde

$$\text{Arc.}\, fg - \text{Arc.}\, Ae = nefg.$$

Puncto ergo g hoc modo definito erit arcuum fg et Ae differentia geometrice assignabilis. Q. E. I.

COROLLARIUM 1

37. Si ergo f ita capiatur, ut sit $1 - neeff = 0$ seu $f = \frac{1}{e\sqrt{n}}$, abscissa $CG = g$ fit infinita ideoque et arcus fg erit infinitus, qui etiam arcum Ae excedere reperitur quantitate infinita $nefg$ ob $g = \infty$. Ut igitur casus, quemadmodum figura repraesentatur, subsistere possit, necesse est, ut capiatur $f < \frac{1}{e\sqrt{n}}$.

COROLLARIUM 2

38. Sin autem sit $f > \frac{1}{e\sqrt{n}}$, fiet g negativum et $\Pi.g$ pariter fiet negativum; unde, si fuerit

$$g = \frac{e\sqrt{(1+ff)(1+nff)} + f\sqrt{(1+ee)(1+nee)}}{neeff-1},$$

habebimus

$$\Pi.e + \Pi.f + \Pi.g = nefg = Ae + Af + Ag.$$

Tres ergo arcus exhiberi possunt Ae, Af et Ag, quorum summa geometrice assignari queat.

COROLLARIUM 3

39. Casus hic, quo summa trium arcuum hyperbolicorum rectificabilis prodiit, eo magis est notatu dignus, quod similis casus in ellipsi locum non habet; ibi enim terni arcus $\Pi.y - \Pi.e - \Pi.x = -ncxy$ (§ 3) nunquam eiusdem signi fieri possunt, propterea quod $necxx$ unitate semper minus existit.

COROLLARIUM 4

40. Horum ternorum arcuum duo inter se fieri possunt aequales; sit enim $f = e$; erit

$$g = \frac{2e\sqrt{(1+ee)(1+nee)}}{ne^4 - 1},$$

unde prodit $2\Pi.e + \Pi.g = neeg$ seu $2\,\text{Arc.}\,Ae + \text{Arc.}\,Ag =$ quantitati geometricae. Si igitur insuper fiat $g = e$, habebitur arcus hyperbolicus, cuius triplum ideoque et ipse ille arcus erit rectificabilis; qui casus cum sit maxime memorabilis, eum in sequente problemate data opera evolvamus.

PROBLEMA 7

41. *In hyperbola a vertice A* (Fig. 5, p. 342) *arcum abscindere Ae, cuius longitudo geometrice assignari queat.*

SOLUTIO

Posito hyperbolae semiaxe transverso $CA = k$ et coniugato $= 1$, ita ut posita abscissa $CE = e$ sit applicata $Ee = k\sqrt{(1+ee)}$, brevitatis gratia autem sit $n = 1 + kk$. Sit ergo $CE = e$ abscissa arcus Ae quaesiti, cuius rectificatio desideratur; quem in finem statuatur in paragrapho praecedenti $g = e$, ut sit

$$e = \frac{2e\sqrt{(1+ee)(1+nee)}}{ne^4 - 1},$$

eritque

$$3\Pi.e = ne^3 \quad \text{seu} \quad \text{Arc.}\,Ae = \frac{1}{3}ne^3$$

ideoque rectificabilis. Abscissa ergo huius arcus $CE = e$ determinari debet

ex hac aequatione $ne^4 - 1 = 2\sqrt{(1+ee)(1+nee)}$, quae abit in hanc

$$nne^8 - 6ne^4 - 4(n+1)ee - 3 = 0.$$

Ad quam resolvendam faciamus $ee = \frac{x}{n}$, ut prodeat

$$x^4 - 6nxx - 4n(n+1)x - 3nn = 0,$$

cuius factores fingantur $(xx + \alpha x + \beta)(xx - \alpha x + \gamma) = 0$; unde comparatione instituta orietur

$$\gamma + \beta = \alpha\alpha - 6n, \quad \gamma - \beta = \frac{-4n(n+1)}{\alpha} \quad \text{et} \quad \beta\gamma = -3nn.$$

Quare cum sit $(\gamma+\beta)^2 - (\gamma-\beta)^2 = 4\beta\gamma = -12nn$, fiet

$$\alpha^4 - 12n\alpha\alpha + 36nn - \frac{16nn(n+1)^2}{\alpha\alpha} = -12nn$$

sive

$$\alpha^6 - 12n\alpha^4 + 48nn\alpha\alpha = 16nn(n+1)^2.$$

Subtrahatur utrinque $64n^3$, ut fiat

$$(\alpha\alpha - 4n)^3 = 16n^2(n-1)^2 \quad \text{seu} \quad \alpha\alpha = 4n + \sqrt[3]{16nn(n-1)^2},$$

ergo

$$\alpha = \sqrt{\left(4n + \sqrt[3]{16nn(n-1)^2}\right)}.$$

Invento nunc valore ipsius α erit porro

$$\beta = \frac{1}{2}\alpha\alpha - 3n + \frac{2n(n+1)}{\alpha} \quad \text{et} \quad \gamma = \frac{1}{2}\alpha\alpha - 3n - \frac{2n(n+1)}{\alpha}$$

et quatuor radices ipsius x erunt

$$x = \pm\frac{1}{2}\alpha \pm \sqrt{\left(3n - \frac{1}{4}\alpha\alpha \pm \frac{2n(n+1)}{\alpha}\right)} = nee,$$

seu cum valor ipsius α tam affirmative quam negative accipi queat, erit

$$e = \sqrt{\left(\frac{\alpha}{2n} \pm \sqrt{\left(\frac{3}{n} - \frac{\alpha\alpha}{4nn} + \frac{2(n+1)}{n\alpha}\right)}\right)}.$$

Hic igitur valor si tribuatur abscissae $CE = e$, erit arcus hyperbolae

$$Ae = \frac{1}{3} ne^3.$$

Q. E. I.

COROLLARIUM 1

42. Si loco unitatis semiaxis coniugatus ponatur $= b$, ut abscissae cuicunque $CP = x$ respondeat applicata $Pp = k\sqrt{\left(1 + \frac{xx}{bb}\right)}$, erit

$$a = \sqrt{\left(4bb(bb + kk) + \sqrt[3]{16b^4k^4(bb + kk)^2}\right)}$$

tumque sumta abscissa

$$CP = x = b\sqrt{\left(\frac{a}{2(bb + kk)} + \sqrt{\left(\frac{2bb}{bb + kk} + \frac{2bb(2bb + kk)}{a(bb + kk)} - \sqrt[3]{\frac{b^4k^4}{4(bb + kk)^4}}\right)}\right)}$$

erit

$$\text{Arc. } Ap = \frac{(bb + kk)x^3}{3b^4}.$$

COROLLARIUM 2

43. Si hyperbola fuerit aequilatera seu $k = b = 1$, poni debet $n = 2$ fietque $a = 2\sqrt{3}$ et arcus rectificabilis Ae abscissa prodit

$$CE = e = \sqrt{\frac{\sqrt{3} + \sqrt{(3 + 2\sqrt{3})}}{2}}$$

et ipsa huius arcus longitudo reperitur

$$Ae = \frac{\sqrt{3} + \sqrt{(3 + 2\sqrt{3})}}{3}\sqrt{\frac{\sqrt{3} + \sqrt{(3 + 2\sqrt{3})}}{2}}.$$

COROLLARIUM 3

44. Si ponatur $4n(n - 1) = s^3$, ut sit $n = \frac{1 + \sqrt{(s^3 + 1)}}{2}$, signa radicalia cubica ex calculo evanescent; prodit enim

$$a = \sqrt{\left(2 + ss + 2\sqrt{(s^3 + 1)}\right)} = \sqrt{(1 - s + ss)} + \sqrt{(1 + s)},$$

unde fit

$$\frac{1+\sqrt{(1+s^3)}}{2}\,ee = \frac{1}{2}\sqrt{(1+s)} + \frac{1}{2}\sqrt{(1-s+ss)}$$

$$\pm\sqrt{\left(1-\frac{1}{4}ss+\sqrt{(1+s^3)}+\left(1-\frac{1}{2}s\right)\sqrt{(1+s)}+\left(1+\frac{1}{2}s\right)\sqrt{(1-s+ss)}\right)}$$

sive

$$ee = \frac{\sqrt{(1+s)}+\sqrt{(1-s+ss)}\pm\sqrt{(4-ss+4\sqrt{(1+s^3)}+2(2-s)\sqrt{(1+s)}+2(2+s)\sqrt{(1-s+ss)})}}{1+\sqrt{(1+s^3)}}.$$

COROLLARIUM 4

45. Pro hyperbola aequilatera, ubi $n=2$, si radicalia per fractiones decimales evolvantur, reperitur $CE = e = 1{,}4619354$ et $Ae = 1{,}4248368\,e$ seu Arc. $Ae = 2{,}0830191$ semiaxe transverso existente $CA = 1$, quos numeros ideo adieci, quo veritas huius rectificationis facilius perspici queat.

COROLLARIUM 5

46. Casus etiam satis simplex prodit, si $s=1$ et $n = \frac{1+\sqrt{2}}{2} = 1+kk$, ita ut sit $k = \sqrt{\frac{\sqrt{2}-1}{2}}$; hinc enim fit

$$ee = \frac{\sqrt{2}+1+\sqrt{(9+6\sqrt{2})}}{1+\sqrt{2}} = 1+\sqrt{3}.$$

Ergo sumta abscissa $CE = \sqrt{(1+\sqrt{3})}$ erit arcus

$$Ae = \frac{(1+\sqrt{2})(1+\sqrt{3})\sqrt{(1+\sqrt{3})}}{6}.$$

In fractionibus decimalibus fit $k = 0{,}45509$, $e = 1{,}65289$ et Arc. $Ae = 1{,}81701$.

COROLLARIUM 6

47. Si sit $s=0$, quo casu fit $n=1$ et $k=0$, hyperbola autem abit in lineam rectam CE, erit $ee=3$ et $e=\sqrt{3}=CE$ arcusque Ae evadit $=\sqrt{3}=CE$, uti natura rei postulat.

PROBLEMA 8

48. *Invenire alios arcus hyperbolicos rectificabiles.*

SOLUTIO

Sumta abscissa $CE = e$ (Fig. 5, p. 342) capiantur aliae duae abscissae $CP = p$ et $CQ = q$, ut sit

$$q = \frac{e\sqrt{(1+pp)(1+npp)} + p\sqrt{(1+ee)(1+nee)}}{1-neepp};$$

erit

$$\Pi.\, q - \Pi.\, p - \Pi.\, e = nepq.$$

Quia ergo

$$\Pi.\, q - \Pi.\, p = \text{Arc.}\, pq \quad \text{et} \quad \Pi.\, e = \text{Arc.}\, Ae,$$

erit

$$\text{Arc.}\, pq = nepq + \text{Arc.}\, Ae.$$

Quodsi igitur abscissae e is tribuatur valor, qui in problemate praecedente est definitus, ita ut arcus Ae sit rectificabilis, hunc scilicet in finem posito

$$\alpha = \sqrt{\left(4n + \sqrt[3]{16nn(n-1)^2}\right)}$$

capiatur

$$e = \sqrt{\left(\frac{\alpha}{2n} + \sqrt{\left(\frac{3}{n} - \frac{\alpha\alpha}{4nn} + \frac{2(n+1)}{n\alpha}\right)}\right)}$$

eritque arcus $Ae = \frac{1}{3}ne^3$. Hinc sumta abscissa p pro lubitu ex superiori formula ita definietur abscissa q, ut prodeat arcus rectificabilis

$$\text{Arc.}\, pq = nepq + \frac{1}{3}ne^3.$$

Verumtamen p ita accipi debet, ut sit $neepp < 1$ seu $p < \frac{1}{e\sqrt{n}}$; cum igitur sit $ne^4 > 1$, capienda est abscissa p minor quam e et quidem oportet sit

$$\frac{1}{p} > \sqrt{\left(\frac{1}{2}\alpha + \sqrt{\left(3n - \frac{1}{4}\alpha\alpha + \frac{2n(n+1)}{\alpha}\right)}\right)}.$$

Dummodo ergo punctum p non capiatur ultra hunc terminum, semper ab eo abscindi potest arcus pq, cuius longitudo geometrice assignari queat. Q. E. I.

COROLLARIUM 1

49. Quodsi capiatur $p = \frac{1}{e\sqrt{n}}$, ob $1 - neepp = 0$ fiet abscissae q valor infinitus ideoque ipse arcus rectificabilis pq erit infinitus.

COROLLARIUM 2

50. In hyperbola ergo aequilatera, ubi $n = 2$ et

$$e = \sqrt{\frac{\sqrt{3} + \sqrt{(3 + 2\sqrt{3})}}{2}},$$

prior abscissa $CP = p$ tam parva accipi debet, ut sit

$$p < \frac{1}{\sqrt{(\sqrt{3} + \sqrt{(3 + 2\sqrt{3})})}} \quad \text{seu} \quad p < 0{,}4836784.$$

Sumta igitur hac abscissa tam parva semper alterum punctum q assignari poterit, ut arcus pq sit rectificabilis.

SCHOLION

51. Insigni hac hyperbolae proprietate, qua reliquis sectionibus conicis antecellit, contentus non immoror investigationi eiusmodi arcuum, quorum differentia sit algebraica vel qui inter se datam teneant rationem, cuiusmodi quaestiones pro ellipsi evolvi; cum enim talia problemata pro hyperbola simili modo resolvi queant, ea, ne lectori sim molestus, data opera praetermitto. Hanc igitur dissertationem finiam comparatione arcuum parabolae cubicalis primariae, cuius rectificationem constat pariter fines Analyseos transgredi.

COMPARATIO ARCUUM PARABOLAE CUBICALIS PRIMARIAE

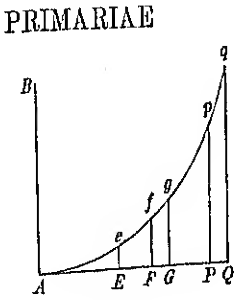

Fig. 6.

52. Sit $Aefg$ (Fig. 6) parabola cubicalis primaria, A eius vertex et $AEFG$ eius tangens in vertice, super qua sumta abscissa quacunque $AP = z$ sit applicata $Pp = \frac{1}{3}z^3$; unde arcus Ap reperitur

$$= \int dz\sqrt{(1 + z^4)} = \int \frac{dz(1 + z^4)}{\sqrt{(1 + z^4)}}.$$

53. Quo igitur formulas nostras huc accommodemus, poni oportet $A=1$, $C=0$, $E=1$, $\mathfrak{A}=1$, $\mathfrak{C}=0$ et $\mathfrak{E}=1$, ita ut sit

$$y=\frac{c\sqrt{(1+x^4)}+x\sqrt{(1+c^4)}}{1-ccxx};$$

quo facto erit

$$\int dx\sqrt{(1+x^4)}-\int dy\sqrt{(1+y^4)}=\text{Const.}-cxy\left(cc+xy\sqrt{(1+c^4)}+\frac{1}{3}ccxxyy\right)$$

sumto tam $\sqrt{A}$ quam c negativo in formulis n° VII et XI expositis.

54. Quodsi ergo tres capiamus abscissas $AE=e$, $AF=f$ et $AG=g$, ita ut sit

$$g=\frac{e\sqrt{(1+f^4)}+f\sqrt{(1+e^4)}}{1-eeff},$$

erit

$$\text{Arc. } Af-\text{Arc. } Ag=-\text{Arc. } Ae-efg\left(ee+fg\sqrt{(1+e^4)}+\frac{1}{3}eeffgg\right)$$

seu

$$\text{Arc. } fg-\text{Arc. } Ae=efg\left(ee+fg\sqrt{(1+e^4)}+\frac{1}{3}eeffgg\right).$$

Dato ergo quovis arcu Ae a dato puncto f abscindi poterit alius arcus fg, ut horum arcuum differentia sit rectificabilis.

55. Si capiantur arcus e et f negativi ita, ut sit $eeff>1$ et

$$g=\frac{e\sqrt{(1+f^4)}+f\sqrt{(1+e^4)}}{eeff-1}$$

et arcus abscissis e, f, g respondentes denotentur per $\Pi.e$, $\Pi.f$, $\Pi.g$, erit

$$\Pi.e+\Pi.f+\Pi.g=efg\left(ee-fg\sqrt{(1+e^4)}+\frac{1}{3}eeffgg\right).$$

Sin autem sit

$$g=\frac{e\sqrt{(1+f^4)}+f\sqrt{(1+e^4)}}{1-eeff},$$

erit

$$\Pi.g-\Pi.f-\Pi.e=efg\left(ee+fg\sqrt{(1+e^4)}+\frac{1}{3}eeffgg\right).$$

56. Cum sit hoc posteriori casu

$$ff + gg = ee + 2fg\sqrt{(1+e^4)} + eeffgg,$$

erit quoque

$$\Pi.g - \Pi.f - \Pi.e = \frac{1}{2}efg\left(ee + ff + gg - \frac{1}{3}eeffgg\right).$$

Casu autem altero pro summa arcuum, quo

$$g = \frac{e\sqrt{(1+f^4)} + f\sqrt{(1+e^4)}}{eeff - 1},$$

erit

$$\Pi.e + \Pi.f + \Pi.g = \frac{1}{2}efg\left(ee + ff + gg - \frac{1}{3}eeffgg\right).$$

PROBLEMA 9

57. *Dato arcu Ae (Fig. 6, p. 349) parabolae cubicalis primariae in eius vertice A terminato ab alio quocunque puncto f abscindere in eadem parabola arcum fg, ita ut horum arcuum differentia $fg - Ae$ sit rectificabilis.*

SOLUTIO

Positis abscissis $AE = e$, $AF = f$, $AG = g$, quarum illae duae dantur, haec vero ita accipiatur, ut sit

$$g = \frac{e\sqrt{(1+f^4)} + f\sqrt{(1+e^4)}}{1 - eeff},$$

eritque horum arcuum differentia

$$\text{Arc. } fg - \text{Arc. } Ae = \frac{1}{2}efg\left(ee + ff + gg - \frac{1}{3}eeffgg\right).$$

Verum cum data sit abscissa e, altera abscissa f ita accipi debet, ut sit $eeff < 1$ seu $f < \frac{1}{e}$, ne abscissa $AG = g$ prodeat negativa. Sin autem detur punctum g, inde reperitur

$$f = \frac{g\sqrt{(1+e^4)} - e\sqrt{(1+g^4)}}{1 - eegg},$$

unde, si g tam fuerit magna, ut sit $eegg > 1$ seu $g > \frac{1}{e}$, erit

$$f = \frac{e\sqrt{(1+g^4)} - g\sqrt{(1+e^4)}}{eegg - 1}$$

simulque necesse est, ut sit $g > e$, ne f fiat negativum. A dato ergo puncto f, siquidem sit $f < \frac{1}{e}$, arcus quaesitus fg in consequentia vergit; a puncto autem g, si sit $g > \frac{1}{e}$ et simul $g > e$, arcus quaesitus fg retro accipietur. Q. E. I.

COROLLARIUM 1

58. Cum sit applicata $Ee = \frac{1}{3}e^3$ seu $AE^3 = 3Ee$, erit parameter huius parabolae $= 3$ ideoque unitas nostra est triens parametri.

COROLLARIUM 2

59. Si ergo sit $e = 1$, abscissa data f seu g vel debet esse minor quam 1 vel maior quam 1; dummodo ergo punctum datum non in e cadat, ab eo semper vel prorsum vel retrorsum arcus quaesito satisfaciens abscindi poterit; prorsum scilicet, si abscissa data minor sit quam e, retrorsum vero, si maior. At si abscissa data esset $= 1$, altera vel infinita vel $= 0$ prodiret.

COROLLARIUM 3

60. Si sit $e > 1$ ideoque $e > \frac{1}{e}$, altera abscissarum f vel g, quae datur, vel minor esse debet quam $\frac{1}{e}$ vel maior quam e; alioquin arcus problemati satisfaciens abscindi nequit, quod ergo usu venit, si abscissa data inter limites e et $\frac{1}{e}$ contineatur.

COROLLARIUM 4

61. Sin autem sit $e < 1$ ideoque $\frac{1}{e} > e$, alteram abscissam datam vel minorem esse oportet quam $\frac{1}{e}$ vel maiorem quam $\frac{1}{e}$; dum ergo non sit aequalis ipsi $\frac{1}{e}$, quo casu arcus quaesitus vel fieret infinitus vel ipsi arcui Ae similis et aequalis, reperietur semper arcus problemati satisfaciens.

COROLLARIUM 5

62. Hoc autem casu, quo $e < 1$, fieri potest, ut a dato puncto f in utramque partem arcus problemati satisfaciens abscindi queat; hoc scilicet evenit, si abscissa data intra limites e et $\frac{1}{e}$ contineatur; tum enim ea tam loco f quam loco g scribi poterit.

COROLLARIUM 6

63. Si arcus fg debeat esse contiguus arcui Ae seu si sit $f = e$, reperietur

$$g = \frac{2e\sqrt{(1+e^4)}}{1-e^4};$$

hoc ergo fieri nequit, nisi sit $e < 1$. Hoc ergo casu erit arcuum differentia

$$\text{Arc. } fg - \text{Arc. } Ae = \frac{2e^5(9 - 2e^4 + e^8)\sqrt{(1+e^4)}}{3(1-e^4)^3}.$$

PROBLEMA 10

64. *Dato in parabola cubicali arcu quocunque fg alium invenire arcum pq, qui illum superet quantitate geometrice assignabili.*

SOLUTIO

Sint abscissae datae $AF = f$, $AG = g$, quaesitae $AP = p$ et $AQ = q$ et in subsidium vocetur arcus Ae, cuius abscissa $AE = e$, sitque

$$g = \frac{e\sqrt{(1+f^4)} + f\sqrt{(1+e^4)}}{1-eeff} \quad \text{et} \quad q = \frac{e\sqrt{(1+p^4)} + p\sqrt{(1+e^4)}}{1-eepp};$$

erit

$$\text{Arc. } fg - \text{Arc. } Ae = \frac{1}{2}efg\left(ee + ff + gg - \frac{1}{3}eeffgg\right) = M$$

et

$$\text{Arc. } pq - \text{Arc. } Ae = \frac{1}{2}epq\left(ee + pp + qq - \frac{1}{3}eeppqq\right) = N,$$

ergo

$$\text{Arc. } pq - \text{Arc. } fg = N - M.$$

Eliminemus autem utrinque e reperieturque

$$e = \frac{g\sqrt{(1+f^4)} - f\sqrt{(1+g^4)}}{1-ffgg} = \frac{q\sqrt{(1+p^4)} - p\sqrt{(1+q^4)}}{1-ppqq},$$

unde, si f, g et p dentur, obtinebitur q hoc modo

$$q = \frac{\left\{\begin{array}{c} g(1-ffgg+ffpp-ggpp)\sqrt{(1+f^4)(1+p^4)} \\ -f(1-ffgg+ggpp-ffpp)\sqrt{(1+g^4)(1+p^4)} \\ +p(1-ffpp-ggpp+ffgg)\sqrt{(1+f^4)(1+g^4)} \\ -2fgp(ff+gg+pp+ffggpp) \end{array}\right\}}{(1-ffgg-ffpp-ggpp)^2 - 4ffggpp(ff+gg+pp)},$$

qui valor quoties non fit negativus, praebebit a dato puncto p arcum pq ab arcu proposito fg geometrice discrepantem. Q. E. I.

COROLLARIUM 1

65. Ambo abscissarum paria ita pendent ab e, ut sit

$$ff+gg = ee(1+ffgg) + 2fg\sqrt{(1+e^4)},$$
$$pp+qq = ee(1+ppqq) + 2pq\sqrt{(1+e^4)},$$

unde reperietur

$$ee = \frac{pq(ff+gg) - fg(pp+qq)}{(pq-fg)(1-fgpq)}$$

et

$$\sqrt{(1+e^4)} = \frac{(pp+qq)(1+ffgg) - (ff+gg)(1+ppqq)}{2(pq-fg)(1-fgpq)}$$

et hinc penitus eliminando e habebitur

$$\big((1-ffgg)(pp+qq) + (1-ppqq)(ff+gg)\big)^2$$
$$= 4(1-fgpq)^2\big((pq-fg)^2 + (ff+gg)(pp+qq)\big)$$

vel

$$\big((1-ffgg)(pp+qq) - (1-ppqq)(ff+gg)\big)^2$$
$$= 4(pq-fg)^2\big((1-fgpq)^2 + (ff+gg)(pp+qq)\big).$$

COROLLARIUM 2

66. Hinc ergo dato quocunque arcu fg infinitis modis alii determinari possunt arcus pq, quorum differentia ab illo fg sit geometrice assignabilis.

Erit autem haec differentia

$$\text{Arc. } pq - \text{Arc. } fg$$

$$= \frac{1}{2}e\Big(ee(pq - fg)\big(1 - \tfrac{1}{3}ppqq - \tfrac{1}{3}fgpq - \tfrac{1}{3}ffgg\big) + pq(pp+qq) - fg(ff+gg)\Big)$$

$$= \frac{e(pq - fg)(ff + gg + pp + qq - \frac{1}{3}pq(pq + 2fg)(ff + gg) - \frac{1}{3}fg(fg + 2pq)(pp + qq))}{2(1 - fgpq)}.$$

COROLLARIUM 3

67. Casus hic duo peculiares considerandi occurrunt, alter quo $pq = fg$, alter quo $fgpq = 1$. Priori casu fit $pp + qq = ff + gg$ ideoque $p = f$ et $q = g$, ita ut arcus pq in ipsum arcum fg incidat eorumque differentia fiat $= 0$. Altero vero casu fit

$$(1 - ffgg)(pp + qq) + \Big(1 - \frac{1}{ffgg}\Big)(ff + gg) = 0 \quad \text{seu} \quad pp + qq = \frac{ff + gg}{ffgg},$$

unde colligitur $p = \frac{1}{g}$ et $q = \frac{1}{f}$, qui est casus a Celeb. IOH. BERNOULLIO[1]) b. m. primum in Actis Lipsiensibus A. 1698 expositus.

COROLLARIUM 4

68. Hoc ergo casu BERNOULLIANO, quo $p = \frac{1}{g}$, $q = \frac{1}{f}$ ac proinde $pq = \frac{1}{fg}$ et $pp + qq = \frac{ff + gg}{ffgg}$, erit arcuum differentia

$$\text{Arc. } pq - \text{Arc. } fg = \frac{e(1 - ffgg)}{6f^3g^3}\big(3(ff + gg)(1 + ffgg) - ee(1 - ffgg)^2\big);$$

at est

$$e(1 - ffgg) = g\sqrt{(1 + f^4)} - f\sqrt{(1 + g^4)},$$

unde colligimus

$$ee(1 - ffgg)^2 = (ff + gg)(1 + ffgg) - 2fg\sqrt{(1 + f^4)(1 + g^4)},$$

quibus valoribus substitutis erit

$$\text{Arc. } pq - \text{Arc. } fg$$

$$= \frac{g\sqrt{(1 + f^4)} - f\sqrt{(1 + g^4)}}{3f^3g^3}\big((ff + gg)(1 + ffgg) + fg\sqrt{(1 + f^4)(1 + g^4)}\big),$$

1) IOH. BERNOULLI, *Theorema universale rectificationi linearum curvarum inserviens. Nova parabolarum proprietas. Cubicalis primariae arcuum mensura etc.* Acta erud. 1698, p. 462; *Opera omnia* t. 1, p. 249. A. K.

quae abit in hanc formam

$$\text{Arc. } pq - \text{Arc. } fg = \frac{(1+f^4)\sqrt{(1+f^4)}}{3f^3} - \frac{(1+g^4)\sqrt{(1+g^4)}}{3g^3},$$

quae est ipsa horum arcuum differentia a Cel. BERNOULLIO exhibita.

SCHOLION

69. Simili modo dato quocunque arcu parabolae cubicalis fg alii arcus inveniri poterunt, qui a duplo vel triplo vel quovis multiplo arcus fg discrepent quantitate algebraica; quin etiam hi arcus ita determinari poterunt, ut differentia evanescat. Hinc ergo proposito arcu quocunque fg alius in eadem parabola assignari poterit, qui arcus istius sit duplus vel triplus vel alius quicunque multiplus. Ex quo vicissim pro lubitu infinitis modis eiusmodi arcus assignare licebit, qui inter se datam teneant rationem. Ut autem duo arcus sint inter se in ratione aequalitatis, alii assignari nequeunt, nisi qui sint inter se similes et aequales. Quod quo clarius appareat, sit

$$fg = m, \quad pq = \mu, \quad ff + gg = n \quad \text{et} \quad pp + qq = \nu;$$

erit primo

$$n = ee(1 + mm) + 2m\sqrt{(1 + e^4)},$$

tum vero

$$\nu = ee(1 + \mu\mu) + 2\mu\sqrt{(1 + e^4)}.$$

Unde ut arcus pq et fg inter se fiant aequales, oportet esse

$$ee(\mu - m)\left(1 - \frac{1}{3}\mu\mu - \frac{1}{3}m\mu - \frac{1}{3}mm\right) + \mu\nu - mn = 0.$$

At pro n et ν illis valoribus substitutis fit

$$\mu\nu - mn = ee(\mu - m)(1 + \mu\mu + m\mu + mm) + 2(\mu - m)(\mu + m)\sqrt{(1 + e^4)},$$

unde debet esse, postquam per $\mu - m$ fuerit divisum,

$$2ee\left(1 + \frac{1}{3}\mu\mu + \frac{1}{3}m\mu + \frac{1}{3}mm\right) + 2(\mu + m)\sqrt{(1 + e^4)} = 0;$$

quae quantitates cum sint omnes affirmativae, solus prior factor $\mu - m = 0$ dabit solutionem eritque $f = p$ et $g = q$. Ad multo illustriora autem progredior ostensurus in hac curva etiam arcus rectificabiles assignari posse.

PROBLEMA 11

70. *In parabola cubicali primaria a vertice A arcum exhibere Ae, cuius longitudo geometrice assignari queat.*

SOLUTIO

Assumtis tribus abscissis $AE=e$, $AF=f$ et $AG=g$ supra vidimus, si sit

$$g=\frac{e\sqrt{(1+f^4)}+f\sqrt{(1+e^4)}}{eeff-1},$$

fore

$$\Pi.e+\Pi.f+\Pi.g=\frac{1}{2}efg\left(ee+ff+gg-\frac{1}{3}eeffgg\right).$$

Statuantur nunc hi tres arcus inter se aequales seu $e=f=g$ eritque

$$e=\frac{2e\sqrt{(1+e^4)}}{e^4-1}\quad\text{seu}\quad e^8-6e^4-3=0$$

hincque

$$e^4=3+2\sqrt{3}.$$

Sumta ergo abscissa

$$AE=e=\sqrt[4]{(3+2\sqrt{3})}$$

erit

$$3\,\text{Arc.}\,Ae=\frac{1}{2}e^6\left(3-\frac{1}{3}e^4\right)=\frac{1}{6}e^6(6-2\sqrt{3})$$

sive

$$\text{Arc.}\,Ae=\frac{1}{9}(3-\sqrt{3})(3+2\sqrt{3})\sqrt[4]{(3+2\sqrt{3})}=\frac{1}{3}(1+\sqrt{3})\sqrt[4]{(3+2\sqrt{3})}.$$

. . . .

FRAGMENTUM
EX ADVERSARIIS MATHEMATICIS DEPROMPTUM[1]

Ex commentatione 819 indicis Enestroemiani
Opera postuma 1, Petropoli 1862, p. 497—502

100.

(I. A. Euler)

PROBLEMA

Pro hyberbola, cuius semiaxis $AC = a$ (Fig. 1), *posito* $AP = x$, $PM = y$ *sit* $ny = \sqrt{(2ax + xx)}$ *et ex* M *ad asymtotam* CN *ducatur* MN *axi parallela; invenire excursum rectae* CN *supra curvam* AM, *quando punctum* M *in infinitum promovetur.*

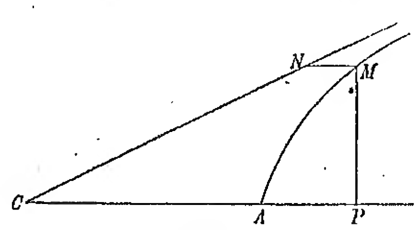

Fig. 1.

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1) In praefatione a Nicolao Fuss minore ad *Opera postuma* scripta legitur p. IV—V: „Praeter scripta postuma ab Eulero ipso elaborata et maximam partem ipsius manu exarata exstant volumina tria, quibus titulus est: *Adversaria mathematica*. His adversariis administri et discipuli Euleri inferre solebant theses quasdam et sententias breves, quas quidem a magistro acceptas ipsi fusius et accuratius explicaverant. Ex his thesibus selectae sunt graviores, quae operibus postumis suo loco inserentur, et primum quidem nonaginta dignae visae sunt, quae typis describerentur. Deinde clarissimus Tschebyscheff, perlustratis iterum dictis voluminibus, invenit alias sex theses, quas addendas esse censuit; has tomus prior exhibet sub Numero XXIII, p. 487—493. In hunc praeterea ex adversariis illatae sunt theses geometricae octo, theses analytici argumenti quatuor et duae ad calculum integralem spectantes; ita ut omnino tomo priori 110 theses ex adversariis depromptae contineantur." — Tomi tres supra commemorati pertinent: tomus I ab a. 1766 usque ad med. Apr. 1775, tomus II inde usque ad Iunium 1779, tomus III inde usque ad mortem Euleri, 1783. A. K.

SOLUTIO

Posito $x = \infty$ fit $ny = x$, hinc

$$\text{tang.}\, ACN = \frac{1}{n} \quad \text{et} \quad \sin. ACN = \frac{1}{\sqrt{(1+nn)}} = \frac{PM}{CN} = \frac{y}{CN},$$

rgo

$$CN = y\sqrt{(1+nn)}.$$

l'um vero habemus $nnyy + aa = (a+x)^2$, ergo

$$x = \sqrt{(nnyy+aa)} - a, \quad \text{unde} \quad dx = \frac{nnydy}{\sqrt{(nnyy+aa)}};$$

hinc arcus

$$AM = \int dy \sqrt{\left(1 + nn - \frac{nnaa}{nnyy+aa}\right)}.$$

Hinc

$$CN - AM = \int dy \left(\sqrt{(nn+1)} - \sqrt{\left(nn + 1 - \frac{nnaa}{nnyy+aa}\right)}\right).$$

Ponatur nunc

$$v = \sqrt{(nn+1)} - \sqrt{\left(nn + 1 - \frac{nnaa}{nnyy+aa}\right)};$$

erit

$$\frac{-nnaa}{nnyy+aa} = -2v\sqrt{(nn+1)} + vv$$

sive

$$\frac{1}{2v\sqrt{(nn+1)} - vv} = \frac{yy}{aa} + \frac{1}{nn},$$

ergo

$$y = \frac{a}{n}\sqrt{\frac{nn - 2v\sqrt{(nn+1)} + vv}{2v\sqrt{(nn+1)} - vv}}.$$

Per logarithmos autem erit

$$2ly - 2la = l\left(nn - 2v\sqrt{(nn+1)} + vv\right) - 2ln - l\left(2v\sqrt{(nn+1)} - vv\right)$$

$$\frac{dy}{y} = \frac{-dv\sqrt{(1+nn)} + vdv}{nn - 2v\sqrt{(nn+1)} + vv} - \frac{dv\sqrt{(nn+1)} - vdv}{2v\sqrt{(nn+1)} - vv};$$

hinc autem vix quicquam concludi poterit.

Ineamus ergo aliam viam. Cum sit

$$CN - AM = \sqrt{(nn+1)}\int dy \left(1 - \sqrt{\left(1 - \frac{nnaa}{nn+1} \cdot \frac{1}{nnyy+aa}\right)}\right),$$

sit

$$\frac{nnaa}{nn+1}\cdot\frac{1}{nnyy+aa}=\cos.\varphi^2;$$

erit $nnyy+aa=\frac{nnaa}{(nn+1)\cos.\varphi^2}$, hinc

$$ny=\frac{a\sqrt{(nn\sin.\varphi^2-\cos.\varphi^2)}}{\cos.\varphi\sqrt{(nn+1)}},$$

ubi casu $y=0$ erit $\cos.\varphi^2=\frac{nn}{nn+1}$ et $\cos.\varphi=\frac{n}{\sqrt{(nn+1)}}$ et $\sin.\varphi=\frac{1}{\sqrt{(nn+1)}}$, $\operatorname{tang}.\varphi=\frac{1}{n}$, hinc $\varphi=ACN$, et pro $y=\infty$ erit $\varphi=90^0$. Ergo integrari debet a $\varphi=ACN$ vel $\operatorname{tang}.\varphi=\frac{1}{n}$ usque ad $\varphi=90^0$ vel $\operatorname{tang}.\varphi=\infty$. Est autem

$$ny=\frac{a\sqrt{(nn\operatorname{tang}.\varphi^2-1)}}{\sqrt{(nn+1)}}.$$

Ponatur $\operatorname{tang}.\varphi=t$ et integrandum a $t=\frac{1}{n}$ usque ad $t=\infty$; at $\sin.\varphi=\frac{t}{\sqrt{(1+tt)}}$. Hinc

$$CN-AM=\sqrt{(1+nn)}\cdot\frac{ann}{n\sqrt{(nn+1)}}\int\frac{tdt}{\sqrt{(nntt-1)}}\left(1-\frac{t}{\sqrt{(1+tt)}}\right)$$

vel

$$CN-AM=\frac{a}{n}\sqrt{(nntt-1)}-\frac{a}{n}\int\frac{nnttdt}{\sqrt{(tt+1)(nntt-1)}}.$$

Est autem

$$\frac{1}{\sqrt{(1+tt)}}=(1+tt)^{-\frac{1}{2}}=\frac{1}{t}-\frac{1}{2}\cdot\frac{1}{t^3}+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{t^5}-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{t^7}+\text{ etc.}$$

Erit

$$CN-AM=na\left(\frac{1}{2}\int\frac{dt}{t\sqrt{(nntt-1)}}-\frac{1\cdot3}{2\cdot4}\int\frac{dt}{t^3\sqrt{(nntt-1)}}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\int\frac{dt}{t^5\sqrt{(nntt-1)}}-\text{etc.}\right).$$

Ubi notandum, si scribatur $t=\frac{1}{u}$, fore

$$\int\frac{dt}{t\sqrt{(nntt-1)}}=\int\frac{-du}{\sqrt{(nn-uu)}}=\text{Arc.}\cos.\frac{u}{n}=\text{Arc.}\cos.\frac{1}{nt}$$

et facto $t=\infty$ erit hoc integrale $=\frac{\pi}{2}$. Deinde

$$\int\frac{dt}{t^3\sqrt{(nntt-1)}}=\int\frac{-uudu}{\sqrt{(nn-uu)}},\quad \int\frac{dt}{t^5\sqrt{(nntt-1)}}=\int\frac{-u^4du}{\sqrt{(nn-uu)}},$$

$$\int\frac{dt}{t^7\sqrt{(nntt-1)}}=\int\frac{-u^6du}{\sqrt{(nn-uu)}}\quad\text{etc.}$$

Fingatur

$$\int\frac{-u^{\lambda+2}du}{\sqrt{(nn-uu)}}=A\int\frac{-u^{\lambda}du}{\sqrt{(nn-uu)}}+Bu^{\lambda+1}\sqrt{(nn-uu)},$$

ubi terminus algebraicus fit $=0$, tam si $u=n$ quam si $u=0$; ergo ob $A=\frac{(\lambda+1)nn}{\lambda+2}$ erit

$$\int\frac{-u^{\lambda+2}du}{\sqrt{(nn-uu)}}=\frac{(\lambda+1)nn}{\lambda+2}\int\frac{-u^{\lambda}du}{\sqrt{(nn-uu)}}.$$

Cum nunc esset

$$\int\frac{-du}{\sqrt{(nn-uu)}}=\frac{\pi}{2},$$

erit

$$\int\frac{-uudu}{\sqrt{(nn-uu)}}=\frac{1}{2}\cdot\frac{\pi}{2}\cdot nn,\quad \int\frac{-u^4du}{\sqrt{(nn-uu)}}=\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi}{2}\cdot n^4,$$

$$\int\frac{-u^6du}{\sqrt{(nn-uu)}}=\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{\pi}{2}\cdot n^6\quad\text{etc.},$$

quamobrem habebimus

$$CN-AM=\frac{\pi}{2}\cdot na\left(\frac{1}{2}-\frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{2}\cdot nn+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1\cdot 3}{2\cdot 4}\cdot n^4-\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot n^6+\text{etc.}\right).$$

Unde excessus in problemate quaesitus $CN-AM$ pro infinito erit

$$\frac{\pi}{2}\cdot na\left(\frac{1}{2}-\frac{1^2}{2^2}\cdot\frac{3}{4}n^2+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}\cdot\frac{5}{6}n^4-\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}\cdot\frac{7}{8}n^6+\text{etc.}\right),$$

quae, si n fuerit unitate minus, valde convergit.

Sequenti autem modo hoc problema elegantius solvetur. Cum sit

$$CN-AM=\sqrt{(1+nn)}\int dy\left(1-\sqrt{\left(1-\frac{nnaa}{1+nn}\cdot\frac{1}{nnyy+aa}\right)}\right),$$

ponatur

$$\frac{nn}{nn+1}=m\quad\text{et}\quad\frac{aa}{nnyy+aa}=uu;$$

erit $\frac{nnyy+aa}{aa}=\frac{1}{uu}$ hincque

$$y=\frac{a}{n}\cdot\frac{\sqrt{(1-uu)}}{u} \quad \text{atque} \quad dy=-\frac{a}{n}\cdot\frac{du}{uu\sqrt{(1-uu)}},$$

ubi pro $y=0$ habemus $u=1$ et pro $y=\infty$ $u=0$. Unde fit

$$CN-AM=\frac{-a}{\sqrt{m}}\int\frac{du(1-\sqrt{(1-muu)})}{uu\sqrt{(1-uu)}},$$

ubi

$$\int\frac{du}{uu\sqrt{(1-uu)}}=\frac{-\sqrt{(1-uu)}}{u}.$$

Pro altero membro

$$\frac{-du}{uu\sqrt{(1-uu)}}\cdot\sqrt{(1-muu)}=\sqrt{(1-muu)}\cdot d.\frac{\sqrt{(1-uu)}}{u}$$

habebimus

$$\int\frac{-du}{uu\sqrt{(1-uu)}}\cdot\sqrt{(1-muu)}=\frac{\sqrt{(1-uu)(1-muu)}}{u}+\int\frac{mdu\sqrt{(1-uu)}}{\sqrt{(1-muu)}}$$

hincque

$$CN-AM=-\frac{a}{\sqrt{m}}\left(-\frac{\sqrt{(1-uu)}}{u}+\frac{\sqrt{(1-uu)(1-muu)}}{u}+\int\frac{mdu\sqrt{(1-uu)}}{\sqrt{(1-muu)}}\right).$$

At si u evanescit, fit $\sqrt{(1-muu)}=1-\frac{1}{2}muu$ et pars integrata sponte evanescit, ita ut iam sit

$$CN-AM=-a\sqrt{m}\int\frac{du\sqrt{(1-uu)}}{\sqrt{(1-muu)}},$$

quod integrari debet a termino $u=1$ usque ad $u=0$; sin autem integremus ab $u=0$ usque ad $u=1$, habebimus

$$CN-AM=a\sqrt{m}\int\frac{du\sqrt{(1-uu)}}{\sqrt{(1-muu)}},$$

cuius valor per rectificationem sectionis conicae assignari potest, uti constat.

Quomadmodum revera est differentia inter asymtotam et arcum hyperbolae, vide Nov. Comm. T. VIII pag. 134 cas. II.[1])

(N. Fuss)

Erit enim
$$CN - AM = aC\sqrt{m} - \frac{am}{m-1}(1 - u\sqrt{m})\Pi,$$
ubi Π est arcus a vertice sumtus sectionis conicae, cuius semiparameter $=1$ et semiaxis transversus $=a$, pro terminis integrationis supra stabilitis.

(I. A. Euler)

Haec formula
$$\int \frac{du\sqrt{(1-uu)}}{\sqrt{(1-muu)}}$$
duplici modo in seriem evolvi potest.

I. Modus. Cum sit
$$(1-muu)^{-\frac{1}{2}} = 1 + \frac{1}{2}muu + \frac{1\cdot 3}{2\cdot 4}m^2u^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}m^3u^6 + \text{ etc.}$$
et
$$\int u^{\lambda+2}du\sqrt{(1-uu)} = \frac{\lambda+1}{\lambda+4}\int u^\lambda du\sqrt{(1-uu)} - \frac{1}{\lambda+4}u^{\lambda+1}(1-uu)^{\frac{3}{2}},$$
ubi postremum membrum ab $u=0$ usque ad $u=1$ sumtum evanescit, quare cum sit
$$\int du\sqrt{(1-uu)} = \frac{\pi}{4},$$
erit
$$\int uudu\sqrt{(1-uu)} = \frac{1}{4}\cdot\frac{\pi}{4}$$
$$\int u^4du\sqrt{(1-uu)} = \frac{1\cdot 3}{4\cdot 6}\cdot\frac{\pi}{4}$$
$$\int u^6du\sqrt{(1-uu)} = \frac{1\cdot 3\cdot 5}{4\cdot 6\cdot 8}\cdot\frac{\pi}{4}$$
etc.

1) L. Euleri Commentatio 273 (indicis Eneströmiani): *Consideratio formularum, quarum integratio per arcus sectionum conicarum absolvi potest*, Novi comment. acad. sc. Petrop. 8 (1760/1), 1763, p. 129; *Leonhardi Euleri Opera omnia*, series I, vol. 20, p. 235, imprimis p. 241. A. K.

Deinde notetur esse $\int d\varphi \cos. 2\lambda\varphi = \frac{1}{2\lambda}\sin. 2\lambda\varphi$, quod casu $\varphi = 90^0$ fit $= 0$; unde patet in evolutione omnes terminos $\sin. 2\lambda\varphi$ continentes omitti posse, unde nostra formula summatoria erit

$$\int d\varphi (1 + k\cos. 2\varphi)^{-\frac{1}{2}}$$
$$= \int d\varphi \left(1 + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{2}kk + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{1\cdot 3}{2\cdot 4}k^4 + \text{etc.}\right)$$

et

$$\int d\varphi \cos. 2\varphi (1 + k\cos. 2\varphi)^{-\frac{1}{2}}$$
$$= \int d\varphi \left(-\frac{1}{2}\cdot\frac{1}{2}k - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1\cdot 3}{2\cdot 4}k^3 - \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10}\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}k^5 - \text{etc.}\right),$$

consequenter

$$CN - AM$$
$$= \frac{1}{2}a\pi\sqrt{\frac{1}{2}}\,k\left(1 - \frac{1}{4}k + \frac{1\cdot 3}{4\cdot 4}kk - \frac{1\cdot 3\cdot 5}{4\cdot 4\cdot 8}k^3 + \frac{1\cdot 3\cdot 5\cdot 7}{4\cdot 4\cdot 8\cdot 8}k^4 - \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{4\cdot 4\cdot 8\cdot 8\cdot 12}k^5 + \text{etc.}\right);$$

casu ergo, quo $n = \infty$, fit $k = 1$, hic vero valor fieri debet $= a$, unde sequitur

$$\frac{2\sqrt{2}}{\pi} = \left(1 - \frac{1}{4} + \frac{1\cdot 3}{4\cdot 4} - \frac{1\cdot 3\cdot 5}{4\cdot 4\cdot 8} + \frac{1\cdot 3\cdot 5\cdot 7}{4\cdot 4\cdot 8\cdot 8} - \text{etc.}\right).$$

Proposita autem vicissim hac serie eius valor ita investigari potest. Fiat $k = zz$ et ponatur

$$s = 1 + \frac{1\cdot 3}{4\cdot 4}z^4 + \frac{1\cdot 3}{4\cdot 4}\cdot\frac{5\cdot 7}{8\cdot 8}z^8 + \text{etc.}$$

et

$$t = \frac{1}{4}z^2 + \frac{1\cdot 3\cdot 5}{4\cdot 4\cdot 8}z^6 + \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{4\cdot 4\cdot 8\cdot 8\cdot 12}z^{10} + \text{etc.},$$

ita ut $s - t$ praebeat nostram seriem. Hinc erit

$$\frac{ds}{dz} = \frac{1\cdot 3}{4}z^3 + \frac{1\cdot 3\cdot 5\cdot 7}{4\cdot 4\cdot 8}z^7 + \text{etc.},$$
$$\frac{d.tz}{dz} = \frac{1\cdot 3}{4}z^2 + \frac{1\cdot 3\cdot 5\cdot 7}{4\cdot 4\cdot 8}z^6 + \text{etc.} = \frac{ds}{zdz},$$

Consequenter fit

$$CN - AM = \frac{\pi a \sqrt{m}}{4}\left(1 + \frac{1\cdot 1}{2\cdot 4}m + \frac{1\cdot 1}{2\cdot 4}\cdot\frac{3\cdot 3}{4\cdot 6}m^2 + \frac{1\cdot 1}{2\cdot 4}\cdot\frac{3\cdot 3}{4\cdot 6}\cdot\frac{5\cdot 5}{6\cdot 8}m^3 + \text{etc.}\right).$$

Hic notandum, si fuerit $m = 1$, fore $CN - AM = a\sqrt{m}\int du$, ut fieri debeat $CN - AM = a$, unde sequitur fore

$$1 = \frac{\pi}{4}\left(1 + \frac{1\cdot 1}{2\cdot 4} + \frac{1\cdot 1}{2\cdot 4}\cdot\frac{3\cdot 3}{4\cdot 6} + \text{etc.}\right)$$

ideoque haec series $= \frac{4}{\pi}$. Alter casus, quo $n = 0$ et $m = 0$, manifesto prodit $CN - AM = 0$.

II. Modus. Ponatur $u = \sin.\varphi$, ita ut integrari oporteat a $\varphi = 0$ usque ad $\varphi = \frac{\pi}{2}$, et habebimus

$$CN - AM = a\sqrt{m}\int\frac{d\varphi\cos.\varphi^2}{\sqrt{(1 - m\sin.\varphi^2)}} = \frac{a\sqrt{m}}{\sqrt{(4 - 2m)}}\int\frac{d\varphi(1 + \cos.2\varphi)}{\sqrt{\left(1 + \frac{m}{2-m}\cos.2\varphi\right)}}.$$

Sit nunc brevitatis gratia $\frac{m}{2-m} = k = \frac{nn}{2+nn}$ et

$$CN - AM = a\sqrt{\frac{1}{2}k}\int d\varphi(1 + \cos.2\varphi)(1 + k\cos.2\varphi)^{-\frac{1}{2}}.$$

Iam vero est

$$(1 + k\cos.2\varphi)^{-\frac{1}{2}} = 1 - \frac{1}{2}k\cos.2\varphi + \frac{1\cdot 3}{2\cdot 4}kk\cos.2\varphi^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}k^3\cos.2\varphi^3 + \text{etc.}$$

Porro notetur esse

$$\cos.2\varphi^2 = \frac{1}{2} + \frac{1}{2}\cos.4\varphi,$$

$$\cos.2\varphi^3 = \frac{3}{4}\cos.2\varphi + \frac{1}{4}\cos.6\varphi,$$

$$\cos.2\varphi^4 = \frac{1\cdot 3}{2\cdot 4} + \frac{1}{2}\cos.4\varphi + \frac{1}{8}\cos.8\varphi,$$

$$\cos.2\varphi^5 = \frac{5}{8}\cos.2\varphi + \frac{5}{16}\cos.6\varphi + \frac{1}{16}\cos.10\varphi,$$

$$\cos.2\varphi^6 = \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} + \text{etc.},$$

$$\cos.2\varphi^7 = 2\cdot\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cos.2\varphi + \text{etc.},$$

$$\cos.2\varphi^8 = \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8} + \text{etc.}$$

Deinde notetur esse $\int d\varphi \cos. 2\lambda\varphi = \frac{1}{2\lambda}\sin. 2\lambda\varphi$, quod casu $\varphi = 90^0$ fit $= 0$; unde patet in evolutione omnes terminos $\sin. 2\lambda\varphi$ continentes omitti posse, unde nostra formula summatoria erit

$$\int d\varphi (1 + k\cos. 2\varphi)^{-\frac{1}{2}}$$
$$= \int d\varphi \left(1 + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{2}kk + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{1\cdot 3}{2\cdot 4}k^4 + \text{etc.}\right)$$

et

$$\int d\varphi \cos. 2\varphi (1 + k\cos. 2\varphi)^{-\frac{1}{2}}$$
$$= \int d\varphi \left(-\frac{1}{2}\cdot\frac{1}{2}k - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1\cdot 3}{2\cdot 4}k^3 - \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10}\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}k^5 - \text{etc.}\right),$$

consequenter

$$CN - AM$$
$$= \frac{1}{2}a\pi\sqrt{\frac{1}{2}}\,k\left(1 - \frac{1}{4}k + \frac{1\cdot 3}{4\cdot 4}kk - \frac{1\cdot 3\cdot 5}{4\cdot 4\cdot 8}k^3 + \frac{1\cdot 3\cdot 5\cdot 7}{4\cdot 4\cdot 8\cdot 8}k^4 - \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{4\cdot 4\cdot 8\cdot 8\cdot 12}k^5 + \text{etc.}\right);$$

casu ergo, quo $n = \infty$, fit $k = 1$, hic vero valor fieri debet $= a$, unde sequitur

$$\frac{2\sqrt{2}}{\pi} = \left(1 - \frac{1}{4} + \frac{1\cdot 3}{4\cdot 4} - \frac{1\cdot 3\cdot 5}{4\cdot 4\cdot 8} + \frac{1\cdot 3\cdot 5\cdot 7}{4\cdot 4\cdot 8\cdot 8} - \text{etc.}\right).$$

Proposita autem vicissim hac serie eius valor ita investigari potest. Fiat $k = zz$ et ponatur

$$s = 1 + \frac{1\cdot 3}{4\cdot 4}z^4 + \frac{1\cdot 3}{4\cdot 4}\cdot\frac{5\cdot 7}{8\cdot 8}z^8 + \text{etc.}$$

et

$$t = \frac{1}{4}z^2 + \frac{1\cdot 3\cdot 5}{4\cdot 4\cdot 8}z^6 + \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{4\cdot 4\cdot 8\cdot 8\cdot 12}z^{10} + \text{etc.},$$

ita ut $s - t$ praebeat nostram seriem. Hinc erit

$$\frac{ds}{dz} = \frac{1\cdot 3}{4}z^3 + \frac{1\cdot 3\cdot 5\cdot 7}{4\cdot 4\cdot 8}z^7 + \text{etc.},$$
$$\frac{d.tz}{dz} = \frac{1\cdot 3}{4}z^2 + \frac{1\cdot 3\cdot 5\cdot 7}{4\cdot 4\cdot 8}z^6 + \text{etc.} = \frac{ds}{zdz},$$

hinc

$$zdt + tdz = \frac{ds}{z}.$$

Porro

$$\frac{d.sz}{dz} = 1 + \frac{1 \cdot 3 \cdot 5}{4 \cdot 4} z^4 + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{4 \cdot 4 \cdot 8 \cdot 8} z^8 + \text{etc.},$$

$$\frac{d.tzz}{dz} = 1 \cdot z^3 + \frac{1 \cdot 3 \cdot 5}{4 \cdot 4} z^7 + \text{ etc. } = \frac{z^3 d.sz}{dz},$$

hinc

$$zzdt + 2tzdz = z^4 ds + sz^3 dz.$$

En ergo has duas aequationes, ex quibus eliminando ds reperitur

$$s = \frac{(1 - z^4)dt}{zdz} + \frac{(2 - z^4)t}{zz},$$

unde

$$ds = \frac{ddt(1 - z^4)}{zdz} - dt\left(\frac{1}{zz} + 3zz\right) + dt\left(\frac{2}{zz} - zz\right) + t\left(-\frac{4}{z^3} - 2z\right)dz$$
$$= zzdt + tzdz;$$

unde resultat haec aequatio

$$0 = zzddt(1 - z^4) + zdzdt(1 - 5z^4) - tdz^2(4 + 3z^4);$$

unde si inventum fuerit t, tunc erit

$$s = \frac{(1 - z^4)dt}{zdz} + \frac{(2 - z^4)t}{zz}.$$

Illa autem aequatio ad differentialem primi gradus reducitur ponendo $t = e^{\int vdz}$, dum erit $dt = e^{\int vdz} vdz$ et $ddt = e^{\int vdz}(dvdz + vvdz^2)$, quibus substitutis reperitur

$$zzdv(1 - z^4) + zzvvdz(1 - z^4) + vzdz(1 - 5z^4) - dz(4 + 3z^4) = 0.$$

Statuatur

$$v = \frac{q}{z(1 - z^4)};$$

erit

$$dv = \frac{dq}{z(1 - z^4)} - \frac{qdz(1 - 5z^4)}{zz(1 - z^4)^2},$$

quibus substitutis nanciscimur

$$dq + \frac{qqdz}{z(1 - z^4)} - \frac{dz(4 + 3z^4)}{z} = 0.$$

hinc

$$zdt + tdz = \frac{ds}{z}.$$

Porro

$$\frac{d.sz}{dz} = 1 + \frac{1\cdot 3\cdot 5}{4\cdot 4}z^4 + \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{4\cdot 4\cdot 8\cdot 8}z^8 + \text{etc.},$$

$$\frac{d.tzz}{dz} = 1\cdot z^3 + \frac{1\cdot 3\cdot 5}{4\cdot 4}z^7 + \text{ etc. } = \frac{z^3 d.sz}{dz},$$

hinc

$$zzdt + 2tzdz = z^4 ds + sz^3 dz.$$

En ergo has duas aequationes, ex quibus eliminando ds reperitur

$$s = \frac{(1-z^4)dt}{zdz} + \frac{(2-z^4)t}{zz},$$

unde

$$ds = \frac{ddt(1-z^4)}{zdz} - dt\left(\frac{1}{zz} + 3zz\right) + dt\left(\frac{2}{zz} - zz\right) + t\left(-\frac{4}{z^3} - 2z\right)dz$$
$$= zzdt + tzdz;$$

unde resultat haec aequatio

$$0 = zzddt(1-z^4) + zdzdt(1-5z^4) - tdz^2(4+3z^4);$$

unde si inventum fuerit t, tunc erit

$$s = \frac{(1-z^4)dt}{zdz} + \frac{(2-z^4)t}{zz}.$$

Illa autem aequatio ad differentialem primi gradus reducitur ponendo $t = e^{\int vdz}$, dum erit $dt = e^{\int vdz}vdz$ et $ddt = e^{\int vdz}(dvdz + vvdz^2)$, quibus substitutis reperitur

$$zzdv(1-z^4) + zzvvdz(1-z^4) + vzdz(1-5z^4) - dz(4+3z^4) = 0.$$

Statuatur

$$v = \frac{q}{z(1-z^4)};$$

erit

$$dv = \frac{dq}{z(1-z^4)} - \frac{qdz(1-5z^4)}{zz(1-z^4)^2},$$

quibus substitutis nanciscimur

$$dq + \frac{qqdz}{z(1-z^4)} - \frac{dz(4+3z^4)}{z} = 0.$$

LEMMA

Notetur haec reductio

$$\int z^{m+n-1}dz(1-z^n)^{k-1} = \frac{m}{m+kn}\int z^{m-1}dz(1-z^n)^{k-1},$$

si integretur a $z=0$ *usque* $z=1$.

ALIA METHODUS EANDEM SERIEM INVESTIGANDI

Quaeratur separatim series

$$s = 1 + \frac{1\cdot 3}{4\cdot 4}kk + \frac{1\cdot 3\cdot 5\cdot 7}{4\cdot 4\cdot 8\cdot 8}k^4 + \text{etc.}$$

et

$$t = \frac{1}{4}k + \frac{1\cdot 3\cdot 5}{4\cdot 4\cdot 8}k^3 + \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{4\cdot 4\cdot 8\cdot 8\cdot 12}k^5 + \text{etc.}$$

Pro priore consideretur formula

$$(1-kkz^4)^{-\frac{1}{4}} = 1 + \frac{1}{4}kkz^4 + \frac{1\cdot 5}{4\cdot 8}k^4z^8 + \frac{1\cdot 5\cdot 9}{4\cdot 8\cdot 12}k^6z^{12} + \text{etc.};$$

hinc erit

$$\int dp(1-kkz^4)^{-\frac{1}{4}} = \int dp + \frac{1}{4}kk\int z^4dp + \frac{1\cdot 5}{4\cdot 8}k^4\int z^8dp + \text{etc.}$$

Nunc fiat

$$\int z^4dp = \frac{3}{4}\int dp \quad \text{et} \quad \int z^8dp = \frac{7}{8}\int z^4dp \quad \text{et} \quad \int z^{12}dp = \frac{11}{12}\int z^8dp;$$

erit

$$s = \frac{\int dp(1-kkz^4)^{-\frac{1}{4}}}{\int dp} = 1 + \frac{1\cdot 3}{4\cdot 4}kk + \frac{1\cdot 3\cdot 5\cdot 7}{4\cdot 4\cdot 8\cdot 8}k^4 + \text{etc.}$$

Ex superiore lemmate habemus

$$\int\frac{z^{m+3}dz}{(1-z^4)^{\frac{3}{4}}} = \frac{m}{m+1}\int\frac{z^{m-1}dz}{(1-z^4)^{\frac{3}{4}}},$$

unde fit

$$\int\frac{z^6dz}{(1-z^4)^{\frac{3}{4}}} = \frac{3}{4}\int\frac{zzdz}{(1-z^4)^{\frac{3}{4}}}, \quad \text{deinde} \quad \int\frac{z^{10}dz}{(1-z^4)^{\frac{3}{4}}} = \frac{7}{8}\int\frac{z^6dz}{(1-z^4)^{\frac{3}{4}}}.$$

Unde patet sumi debere

$$dp = \frac{zzdz}{(1-z^4)^{\frac{3}{4}}};$$

consequenter erit

$$s = \int \frac{zzdz}{(1-z^4)^{\frac{3}{4}}(1-kkz^4)^{\frac{1}{4}}} : \int \frac{zzdz}{(1-z^4)^{\frac{3}{4}}}.$$

Pro altera serie

$$t = \frac{1}{4}k + \frac{1\cdot 3\cdot 5}{4\cdot 4\cdot 8}k^3 + \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{4\cdot 4\cdot 8\cdot 8\cdot 12}k^5 + \text{ etc.}$$

consideretur

$$\frac{(1-kkz^4)^{-\frac{1}{4}}-1}{kzz} = \frac{1}{4}kz^2 + \frac{1\cdot 5}{4\cdot 8}k^3z^6 + \frac{1\cdot 5\cdot 9}{4\cdot 8\cdot 12}k^5z^{10} + \text{ etc.}$$

Fiat

$$\int z^6dp = \frac{3}{4}\int z^2dp, \quad \int z^{10}dp = \frac{7}{8}\int z^6dp \quad \text{etc.},$$

hinc

$$dp = \frac{dz}{(1-z^4)^{\frac{3}{4}}},$$

unde sequitur

$$t = \int \frac{dz\left((1-kkz^4)^{-\frac{1}{4}}-1\right)}{kzz(1-z^4)^{\frac{3}{4}}} : \int \frac{dz}{(1-z^4)^{\frac{3}{4}}}.$$

Hinc autem neutiquam patet, quomodo haec series commodius exprimi possit.

Adversaria mathematica, t. I, p. 258—266.

FRAGMENTA NOVA EX ADVERSARIIS MATHEMATICIS DEPROMPTA[1]

Ex manuscriptis academiae scientiarum Petropolitanae nunc primum edita

I

Superior rectificatio ellipsis[2]) facilius hoc modo expeditur. Sit ACB (Fig. 1) quadrans ellipticus, $AC = a$, $BC = b$ et ACD quadrans circuli radii $AC = a$; ducta applicata XYV et radio CV sit angulus $ACV = \varphi$; erit $CX = a \cos. \varphi = x$, $XV = a \sin. \varphi$ et $XY = y = b \sin. \varphi$, ergo

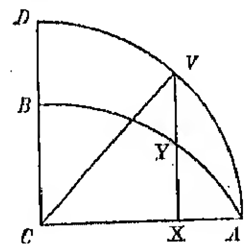

Fig. 1.

$$dx = -a d\varphi \sin. \varphi, \quad dy = b d\varphi \cos. \varphi,^{3)}$$

unde $\sqrt{(dx^2 + dy^2)} = d\varphi \sqrt{(aa \sin. \varphi^2 + bb \cos. \varphi^2)}$ et arcus

$$AY = \int d\varphi \sqrt{(aa \sin. \varphi^2 + bb \cos. \varphi^2)} = \int d\varphi \sqrt{\left(\frac{aa+bb}{2} - \frac{aa-bb}{2} \cos. 2\varphi\right)};$$

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1) Vide notam p. 358. Maximam partem quaestionum in *Adversariis mathematicis* ad theoriam integralium ellipticorum pertinentium commentationes iam editae continent. Quae praeterea publicatione digna videbantur, volumine 20 iam edito selecta sunt ideoque haec fragmenta nova in praefatione commemorari non poterant. A. K.

2) Vide L. Euleri Commentationem 448 (indicis Eneströmiani): *Nova series infinita maxime convergens perimetrum ellipsis exprimens*, Novi comment. acad. sc. Petrop. 18 (1773), 1774, p. 71; *Leonhardi Euleri Opera omnia*, series I, vol. 20, p. 357. A. K.

3) In manuscriptis Euleri signum differentiationis ubique est ∂ (quod omnino in scribendo usurpat), non d. Quia autem in *Operibus postumis*, quibus continentur omnia fragmenta iam edita, ubique signum d usurpatum est, etiam in his fragmentis edendis hoc signo usi sumus. A. K.

hinc, si ponatur $aa+bb=cc$ et $\frac{aa-bb}{aa+bb}=n$, fiet arcus

$$AY=\frac{c}{\sqrt{2}}\int d\varphi\sqrt{(1-n\cos.2\varphi)}.$$

At

$$\sqrt{(1-n\cos.2\varphi)}$$

$$=1-\frac{1}{2}n\cos.2\varphi-\frac{1\cdot1}{2\cdot4}nn\cos.2\varphi^2-\frac{1\cdot1\cdot3}{2\cdot4\cdot6}n^3\cos.2\varphi^3-\frac{1\cdot1\cdot3\cdot5}{2\cdot4\cdot6\cdot8}n^4\cos.2\varphi^4-\text{etc.}$$

Iam

$$\int d\varphi\cos.2\varphi=\frac{1}{2}\sin.2\varphi,$$

quod pro toto quadrante, ubi $\varphi=90^0$, fit $=0$; deinde

$$\int d\varphi\cos.2\varphi^2=\frac{1}{2}\int d\varphi(1+\cos.4\varphi)=\frac{1}{2}\varphi+\frac{1}{8}\sin.4\varphi,$$

si $\varphi=90^0$, fit $=\frac{\pi}{4}$; porro

$$\int d\varphi\cos.2\varphi^3=0,\quad \int d\varphi\cos.2\varphi^4=\frac{1\cdot3}{2\cdot4}\cdot\frac{\pi}{2},$$

$$\int d\varphi\cos.2\varphi^5=0,\quad \int d\varphi\cos.2\varphi^6=\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{\pi}{2}$$

etc.,

si $\varphi=90^0$. Consequenter quadrans ellipticus

$$AYB=\frac{c}{\sqrt{2}}\left(\frac{\pi}{2}-\frac{1\cdot1}{2\cdot4}n^2\frac{1}{2}\cdot\frac{\pi}{2}-\frac{1\cdot1\cdot3\cdot5}{2\cdot4\cdot6\cdot8}n^4\frac{1\cdot3}{2\cdot4}\cdot\frac{\pi}{2}-\frac{1\cdot1\cdot3\cdot5\cdot7\cdot9}{2\cdot4\cdot6\cdot8\cdot10\cdot12}n^6\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{\pi}{2}-\text{etc.}\right);$$

sit coefficiens secundi termini α, tertii β, quarti γ, quinti δ etc.; erit

$$\alpha=\frac{1\cdot1}{2\cdot4}\cdot\frac{1}{2}=\frac{1\cdot1}{4\cdot4},\quad \frac{\beta}{\alpha}=\frac{3\cdot5}{6\cdot8}\cdot\frac{3}{4}=\frac{3\cdot5}{8\cdot8},\quad \frac{\gamma}{\beta}=\frac{7\cdot9}{10\cdot12}\cdot\frac{5}{6}=\frac{7\cdot9}{12\cdot12}\quad\text{etc.},$$

ergo

$$AYB=\frac{\pi}{2\sqrt{2}}\left(1-\frac{1\cdot1}{4\cdot4}nn-\frac{1\cdot1\cdot3\cdot5}{4\cdot4\cdot8\cdot8}n^4-\frac{1\cdot1\cdot3\cdot5\cdot7\cdot9}{4\cdot4\cdot8\cdot8\cdot12\cdot12}n^6-\text{etc.}\right).$$

Circa seriem modo pro perimetro ellipsis inventam posito $n = xx$ si statuatur

$$s = 1 - \frac{1 \cdot 1}{4 \cdot 4} x^4 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{4 \cdot 4 \cdot 8 \cdot 8} x^8 - \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{4 \cdot 4 \cdot 8 \cdot 8 \cdot 12 \cdot 12} x^{12} - \text{etc.},$$

inter s et x haec reperitur aequatio differentialis secundi gradus

$$\frac{xxdds}{dx^2} + \frac{xds}{dx} + \frac{x^4 s}{1 - x^4} = 0;$$

nam multiplicando per $1 - x^4$ erit

$$\frac{xxdds}{dx^2}(1 - x^4) + \frac{xds}{dx}(1 - x^4) + x^4 s = 0,$$

ac si fingatur

$$s = 1 + Ax^4 + Bx^8 + Cx^{12} + Dx^{16} + \text{etc.},$$

erit

$$\frac{ds}{dx} = 4Ax^3 + 8Bx^7 + 12Cx^{11} + 16Dx^{15} + \text{etc.},$$

$$\frac{dds}{dx^2} = 3 \cdot 4Ax^2 + 7 \cdot 8Bx^6 + 11 \cdot 12Cx^{10} + 15 \cdot 16Dx^{14} + \text{etc.},$$

ergo

$$\begin{array}{rcccccccccc}
\dfrac{xxdds}{dx^2} & = & & & 3 \cdot 4Ax^4 & + & 7 \cdot 8Bx^8 & + & 11 \cdot 12Cx^{12} & + & 15 \cdot 16Dx^{16} + \text{etc}, \\
-\dfrac{xxdds}{dx^2}x^4 & = & & & & - & 3 \cdot 4A & - & 7 \cdot 8B & - & 11 \cdot 12C \\
\dfrac{xds}{dx} & = & & & 4A & + & 8B & + & 12C & + & 16D \\
-\dfrac{xds}{dx}x^4 & = & & & & - & 4A & - & 8B & - & 12C \\
x^4 s & = & & & 1 & + & A & + & B & + & C
\end{array}$$

unde fit

$$\begin{aligned}
4 \cdot 4A + 1 &= 0, & A &= -\frac{1}{4 \cdot 4}, \\
8 \cdot 8B - 3 \cdot 5A &= 0, & B &= \frac{3 \cdot 5}{8 \cdot 8} A, \\
12 \cdot 12C - 7 \cdot 9B &= 0, & C &= \frac{7 \cdot 9}{12 \cdot 12} B, \\
16 \cdot 16D - 11 \cdot 13C &= 0, & D &= \frac{11 \cdot 13}{16 \cdot 16} C
\end{aligned}$$

etc.

II

THEOREMA

Haec aequatio differentialis

$$\frac{dx}{\sqrt{(\alpha+\beta x+\gamma xx+\delta x^3+\varepsilon x^4+\zeta x^5)}}=\frac{dy}{\sqrt{(\alpha+\beta y+\gamma yy+\delta y^3+\varepsilon y^4+\zeta y^5)}}$$

integrale algebraicum habere nequit, nisi sit $\zeta=0$; id quod unico casu speciali ostendisse sufficit.[1])

Consideremus ergo hoc exemplum

$$\frac{dx}{\sqrt{(x-2x^3+x^5)}}=\frac{dy}{\sqrt{(y-2y^3+y^5)}}$$

ac posito $x=pp$ et $y=qq$ haec aequatio fiet

$$\frac{dp}{1-p^4}=\frac{dq}{1-q^4}$$

sive

$$\frac{dp}{1-pp}+\frac{dp}{1+pp}=\frac{dq}{1-qq}+\frac{dq}{1+qq},$$

cuius integrale est

$$\frac{1}{2}\,l\frac{1+p}{1-p}+\text{Arc. tang. }p=\frac{1}{2}\,l\frac{1+q}{1-q}+\text{Arc. tang. }q+C;$$

quod quia duplicis generis quantitates transcendentes continet, relatio inter p et q algebraica esse nequit.

Consideremus etiam

$$\frac{dx}{\sqrt{(x^5-x^4)}}=\frac{dy}{\sqrt{(y^5-y^4)}},$$

quod transit in hoc

$$\frac{dx}{xx\sqrt{(x-1)}}=\frac{dy}{yy\sqrt{(y-1)}}.$$

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1) Vide etiam *Institutionum calculi integralis* vol. I, § 640, LEONHARDI EULERI *Opera omnia*, series I, vol. 11, p. 414. A. K.

Iam differentietur eritque

$$dy(a+bx+cxx) = dx(g+hx-by-2cxy).$$

Erat autem

$$\sqrt{((g-by)^2+(f-2ay)(2cy-h))} = x(2cy-h)-g+by,$$

ita ut iam sit

$$dy(a+bx+cxx) = dx\sqrt{((g-by)^2+(f-2ay)(2cy-h))}$$

sive

$$\frac{dx}{a+bx+cxx} = \frac{dy}{\sqrt{(gg-fh+2(ah+cf-bg)y+(bb-4ac)yy)}},$$

cuius ergo integrale est ipsa aequatio algebraica proposita.

Vicissim ergo si proponatur haec aequatio differentialis

$$\frac{dx}{a+bx+cxx} = \frac{dy}{\sqrt{(kyy+2my+n)}},$$

algebraice integrari poterit, si modo fuerit $k=bb-4ac$; tum enim ponatur $m=ah+cf-bg$ et $n=gg-fh$, unde datis m et n quaeri debent f, g, h. Ex priori autem fit $g=\frac{ah+cf-m}{b}$, ex altera vero $g=\sqrt{(n+fh)}$, ex quo fit

$$(ah+cf)^2-2m(ah+cf)+mm=bb(n+fh).$$

Ponatur brevitatis gratia $ah+cf=p$ et $fh=q$, unde fit $g=\frac{p-m}{b}$, deinde erit $(ah-cf)^2=pp-4acq$, ergo

$$h=\frac{p+\sqrt{(pp-4acq)}}{2a} \quad \text{et} \quad f=\frac{p-\sqrt{(pp-4acq)}}{2c};$$

tum vero erit $pp-2mp+mm=bb(n+q)$ hincque

$$q=\frac{pp-2mp+mm-bbn}{bb},$$

quantitas autem p manet indeterminata ideoque refert constantem arbitrariam per integrationem ingressam. Constat autem prioris formulae integrale esse algebraicum, si $bb=4ac$; at si $bb>4ac$, integrale exprimitur per logarithmos, at si $bb<4ac$, per arcum circularem.

Sit $a=1$, $b=0$ et $c=1$, inde $k=-4$; ergo huius aequationis differentialis

$$\frac{dx}{1+xx}=\frac{dy}{\sqrt{(n+2my-4yy)}}$$

integrale algebraice exprimi poterit. ... Hic ob $b=0$ littera g per p definiri nequit ... Reperitur autem tandem

$$h=m-f \quad \text{et} \quad g=\sqrt{(n+fh)};$$

tum vero integrale completum est

$$2y(1+xx)=f+2gx+hxx$$

sicque f est constans arbitraria.

Ille autem calculus ita commodius institui potest. Cum sit

$$ah+cf=bg+m \quad \text{et} \quad fh=gg-n,$$

erit

$$ah-cf=\sqrt{((bg+m)^2-4ac(gg-n))}$$
$$=\sqrt{((bb-4ac)gg+2bgm+mm+4acn)},$$

unde f et h facile definiuntur, et littera g est quantitas arbitraria.

Hinc in exemplo allato, quo $a=1$, $b=0$ et $c=1$, erit

$$h+f=m \quad \text{et} \quad h-f=\sqrt{(mm+4n-4gg)}$$

ideoque

$$h=\frac{1}{2}m+\frac{1}{2}\sqrt{(mm+4n-gg)} \quad \text{et} \quad f=\frac{1}{2}m-\frac{1}{2}\sqrt{(mm+4n-gg)}$$

sicque huius aequationis

$$\frac{dx}{1+xx}=\frac{dy}{\sqrt{(n+2my-4yy)}}$$

integrale completum erit

$$4y(1+xx)=m-\sqrt{(mm+4n-gg)}+4gx+mxx+xx\sqrt{(mm+4n-gg)}.$$

PROBLEMA

Invenire conditiones, sub quibus huius aequationis

$$\frac{dx}{Ax^2+2Bx+C}+\frac{dy}{\mathfrak{A}y^2+2\mathfrak{B}y+\mathfrak{C}}=0$$

integrale completum algebraice exhiberi potest.

SOLUTIO

Fingatur aequatio canonica

$$(x+f)(y+g)=h;$$

erit differentiando

$$dx(y+g)+dy(x+f)=0 \quad \text{ideoque} \quad \frac{dx}{x+f}+\frac{dy}{y+g}=0.$$

Multiplicetur per $\frac{1}{m(x+f)+n(y+g)+2k}$ et ob $(x+f)(y+g)=h$ orietur haec aequatio

$$\frac{dx}{m(x+f)^2+nh+2k(x+f)}+\frac{dy}{n(y+g)^2+mh+2k(y+g)}=0,$$

quae habet ipsam formam propositam, eritque

$$A=m, \quad B=mf+k, \quad C=mff+2kf+nh,$$
$$\mathfrak{A}=n, \quad \mathfrak{B}=ng+k, \quad \mathfrak{C}=ngg+2gk+mh.$$

Cum igitur sit $m=A$ et $n=\mathfrak{A}$, ex aequationibus secundis quaerantur litterae f et g eritque $f=\frac{B-k}{A}$ et $g=\frac{\mathfrak{B}-k}{\mathfrak{A}}$. Hi valores substituantur in tertiis, quae erunt

$$C=\frac{BB-kk}{A}+\mathfrak{A}h \quad \text{et} \quad \mathfrak{C}=\frac{\mathfrak{B}\mathfrak{B}-kk}{\mathfrak{A}}+Ah,$$

unde fit $AC-\mathfrak{A}\mathfrak{C}=BB-\mathfrak{B}\mathfrak{B}$, unde definitur haec conditio

$$AC-BB=\mathfrak{A}\mathfrak{C}-\mathfrak{B}\mathfrak{B}.$$

Praeterea vero erit

$$h=\frac{AC-BB+kk}{A\mathfrak{A}} \quad \text{seu} \quad h=\frac{\mathfrak{A}\mathfrak{C}-\mathfrak{B}\mathfrak{B}+kk}{\mathfrak{A}A},$$

tum

$$f=\frac{B-k}{A} \quad \text{et} \quad g=\frac{\mathfrak{B}-k}{\mathfrak{A}}.$$

PROBLEMA

Invenire conditiones, sub quibus huius aequationis

$$\frac{dx}{Ax^2+2Bx+C}+\frac{dy}{\mathfrak{A}y^2+2\mathfrak{B}y+\mathfrak{C}}=0$$

integrale completum algebraice exhiberi potest.

SOLUTIO

Fingatur aequatio canonica

$$(x+f)(y+g)=h;$$

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Multiplicetur per $\frac{1}{m(x+f)+n(y+g)+2k}$ et ob $(x+f)(y+g)=h$ orietur haec aequatio

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$$C=\frac{BB-kk}{A}+\mathfrak{A}h \quad \text{et} \quad \mathfrak{C}=\frac{\mathfrak{B}\mathfrak{B}-kk}{\mathfrak{A}}+Ah,$$

unde fit $AC-\mathfrak{A}\mathfrak{C}=BB-\mathfrak{B}\mathfrak{B}$, unde definitur haec conditio

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Praeterea vero erit

$$h=\frac{AC-BB+kk}{A\mathfrak{A}} \quad \text{seu} \quad h=\frac{\mathfrak{A}\mathfrak{C}-\mathfrak{B}\mathfrak{B}+kk}{\mathfrak{A}A},$$

tum

$$f=\frac{B-k}{A} \quad \text{et} \quad g=\frac{\mathfrak{B}-k}{\mathfrak{A}}.$$

huiusque formae differentiale ipsam propositam debet producere, quod facienti patebit. Methodo solita autem integrale primae partis est Arc. tang. x; pro altera parte ponatur $y + \frac{3}{2} = z$ sive $y = z - \frac{3}{2}$ eritque membrum

$$\frac{dz}{2zz+\frac{1}{2}} = \frac{2dz}{4zz+1} = \text{Arc. tang. } 2z = \text{Arc. tang. } (2y+3).$$

Erit ergo

$$\text{Arc. tang. } x + \text{Arc. tang. } (2y+3) = \text{Arc. tang. } a$$

ideoque

$$\frac{x+2y+3}{1-2xy-3x} = a,$$

ubi $a = -\frac{1}{k}$.

Adversaria mathematica, t. II, p. 178—179, 185.